INVARIANTS OF MEASURE AND CATEGORY

TOMEK BARTOSZYNSKI

1. Introduction

The purpose of this chapter is to discuss various results concerning the relationship between measure and category. We are mostly interested in set-theoretic properties of these ideals, particularly, their cardinal characteristics. This is a very large area, and it was necessary to make some choices. We decided to present several new results and new approaches to old problems. In most cases we do not present the optimal result, but a simpler theorem that still carries most of the weight of that original result. For example, we construct Borel morphisms in the Cichoń diagram while continuous ones can be constructed. We believe however that the reader should have no problems upgrading the material presented here to the current state of the art. The standard reference for this subject is [8], and this chapter updates it as most of the material presented here was proved after [8] was published.

Measure and category have been studied for about a century. The beautiful book [31] contains a lot of classical results, mostly from analysis and topology, that involve these notions. The role played by Lebesgue measure and the Baire category in these results is more or less identical. There are, of course, theorems indicating lack of complete symmetry but they do not seem very significant. For example, Kuratowski's theorem (cf. Theorem 3.7) asserts that for every Borel function $f : {}^\omega 2 \longrightarrow {}^\omega 2$ there exists a meager set $F \subseteq {}^\omega 2$ such that $f{\restriction}({}^\omega 2 \setminus F)$ is continuous. The dual proposition stating that for every Borel function $f : {}^\omega 2 \longrightarrow {}^\omega 2$ there exists a measure one set $G \subseteq {}^\omega 2$ such that $f{\restriction}G$ is continuous is false. We only have a theorem of Luzin which guarantees that a such G's can have measure arbitrarily close to one.

The last 15 years have brought a wealth of results indicating that hypotheses relating to measure are often stronger than the analogous ones relating to category. This chapter contains several examples of this phenomenon. Before we delve into this subject let us give a little historical background. The first result of this kind is due to Shelah [43]. He showed that

- If all projective sets are measurable then there exists an inner model with an inaccessible cardinal.
- It is consistent with ZFC that all projective sets have the property of Baire.

In 1984 Bartoszynski [2] and Raisonnier and Stern in their [35] showed that additivity of measure is not greater than additivity of category, while Miller [29] showed that it can be strictly smaller. In subsequent years several more results of that kind were found. Let us mention one more (cf. [5]) concerning filters on ω (treated as subsets of ${}^\omega 2$):

Author partially supported by NSF grant DMS 95-05375 and Alexander von Humboldt Foundation.

- There exists a measurable filter that does not have the Baire property. In fact, every filter that has measure zero can be extended to a measure zero filter that does not have the Baire property.
- It is consistent with ZFC that every filter that has the Baire property is measurable.

All these results as well as many others concerning measurability and the Baire property of projective sets, connections with forcing and others can be found in [8].

2. Tukey connections

The starting point for our considerations is the following list of cardinal invariants of an ideal. For an ideal $\mathcal{J}$ of subsets of a set X define

1. $\mathsf{add}(\mathcal{J}) = \min\{|\mathcal{A}| : \mathcal{A} \subseteq \mathcal{J} \text{ and } \bigcup \mathcal{A} \notin \mathcal{J}\}$,
2. $\mathsf{cov}(\mathcal{J}) = \min\{|\mathcal{A}| : \mathcal{A} \subseteq \mathcal{J} \text{ and } \bigcup \mathcal{A} = X\}$,
3. $\mathsf{non}(\mathcal{J}) = \min\{|Y| : Y \subseteq X \text{ and } Y \notin \mathcal{J}\}$,
4. $\mathsf{cof}(\mathcal{J}) = \min\{|\mathcal{A}| : \mathcal{A} \subseteq \mathcal{J} \text{ and } \forall B \in \mathcal{J}\ \exists A \in \mathcal{A}\ B \subseteq A\}$.

Definition 2.1. *Suppose that $\mathcal{P}$ and $\mathcal{Q}$ are partial orderings. We say that $\mathcal{P} \preceq \mathcal{Q}$ if there is function $f : \mathcal{P} \longrightarrow \mathcal{Q}$ such that for every bounded set $X \subseteq \mathcal{Q}$, $f^{-1}(X)$ is bounded in $\mathcal{P}$. Such a function f is called Tukey embedding. Define $\mathcal{P} \equiv \mathcal{Q}$ if $\mathcal{P} \preceq \mathcal{Q}$ and $\mathcal{Q} \preceq \mathcal{P}$.*

Note that if $f : \mathcal{P} \longrightarrow \mathcal{Q}$ is a Tukey embedding then there is an associated function $f^\star : \mathcal{Q} \longrightarrow \mathcal{P}$ defined such that $f^\star(q)$ is a bound of the set $f^{-1}\big(\{(p) : p \leq q\}\big)$. Observe that f maps every set unbounded in $\mathcal{P}$ onto a set unbounded in $\mathcal{Q}$ and $f^\star$ maps every set cofinal in $\mathcal{Q}$ onto a set cofinal in $\mathcal{P}$.

Lemma 2.2. *Suppose that $\mathcal{I}$ and $\mathcal{J}$ are two ideals. If $\mathcal{I} \preceq \mathcal{J}$, then $\mathsf{add}(\mathcal{I}) \geq \mathsf{add}(\mathcal{J})$ and $\mathsf{cof}(\mathcal{I}) \leq \mathsf{cof}(\mathcal{J})$.*

Proof. Suppose that $f : \mathcal{I} \longrightarrow \mathcal{J}$ is a Tukey function.

Let $\mathcal{A} \subseteq \mathcal{I}$ be a family of size $< \mathsf{add}(\mathcal{J})$. Find a set $B \in \mathcal{J}$ such that $\bigcup_{A \in \mathcal{A}} f(A) \subseteq B$. It follows that $\bigcup \mathcal{A} \subseteq f^\star(B)$.

Similarly, if $\mathcal{B} \subseteq \mathcal{J}$ is a basis for $\mathcal{J}$, then $\{f^\star(B) : B \in \mathcal{B}\}$ is a basis for $\mathcal{I}$. □

We will need a slightly stronger definition which will encompass both cardinal invariants and Tukey embeddings.

Definition 2.3. *Suppose that $\mathbf{A} = (A_-, A_+, A)$, where A is a binary relation between A_- and A_+. Let*

$$\mathfrak{d}(\mathbf{A}) = \{Z : Z \subseteq A_+ \ \&\ \forall x \in A_-\ \exists z \in Z\ A(x,z)\}.$$

$$\mathfrak{b}(\mathbf{A}) = \{Z : Z \subseteq A_- \ \&\ \forall y \in A_+\ \exists z \in Z\ \neg A(z,y)\}.$$

$$\|\mathbf{A}\| = \min\{|Z| : Z \in \mathfrak{d}(\mathbf{A})\}.$$

Define $\mathbf{A}^\perp = (A_+, A_-, A^\perp)$, where $A^\perp = \{(z,x) : \neg A(x,z)\}$. Note that $\mathfrak{b}(\mathbf{A}) = \mathfrak{d}(\mathbf{A}^\perp)$.

Note that $\|\mathbf{A}\|$ is the smallest size of the "dominating" family in A_+ and $\|\mathbf{A}^\perp\|$ is the smallest size of the "unbounded" family in A_-. Virtually all cardinal characteristics of the continuum can be expressed in this framework. For two ideals $\mathcal{I} \subseteq \mathcal{J}$ of subsets of X we have:

- $\mathsf{cof}(\mathcal{J}) = \|(\mathcal{J}, \mathcal{J}, \subseteq)\|$,

- $\mathsf{add}(\mathcal{J}) = \|(\mathcal{J}, \mathcal{J}, \subseteq)^\perp\| = \|(\mathcal{J}, \mathcal{J}, \not\supseteq)\|$,
- $\mathsf{cov}(\mathcal{J}) = \|(X, \mathcal{J}, \in)\|$,
- $\mathsf{non}(\mathcal{J}) = \|(X, \mathcal{J}, \in)^\perp\| = \|(\mathcal{J}, X, \not\ni)\|$,
- $\mathfrak{d} = \|({}^\omega\omega, {}^\omega\omega, \leq^\star)\|$,
- $\mathfrak{b} = \|({}^\omega\omega, {}^\omega\omega, \leq^\star)^\perp\| = \|({}^\omega\omega, {}^\omega\omega, \not\geq^\star)\|$, where for $f, g \in {}^\omega\omega$ we define $f \leq^\star g$ if $f(n) \leq g(n)$ for all but finitely many $n \in \omega$.

The notion of Tukey embedding generalizes to the following:

Definition 2.4. *A morphism φ between $\mathbf{A}$ and $\mathbf{B}$ is a pair of functions $\varphi_- : A_- \longrightarrow B_-$ and $\varphi_+ : B_+ \longrightarrow A_+$ such that for each $a \in A_-$ and $b \in B_+$,*

$$A\big(a, \varphi_+(b)\big), \text{ whenever } B\big(\varphi_-(a), b\big).$$

If there is a morphism between $\mathbf{A}$ and $\mathbf{B}$, we say that $\mathbf{A} \preceq \mathbf{B}$.

Note that if a pair of functions $f, f^\star$ witnesses that $\mathcal{P} \preceq \mathcal{Q}$, then $\varphi = (f, f^\star)$ is a morphism between $(\mathcal{P}, \mathcal{P}, \leq)$ and $(\mathcal{Q}, \mathcal{Q}, \leq)$.

Lemma 2.5. 1. $\mathbf{A} \leq \mathbf{B} \iff \mathbf{A}^\perp \geq \mathbf{B}^\perp$,
2. *If $\mathbf{A} \leq \mathbf{B}$, then $\|\mathbf{A}\| \leq \|\mathbf{B}\|$ and $\|\mathbf{A}^\perp\| \geq \|\mathbf{B}^\perp\|$.*

Proof. (1) If $\varphi = (\varphi_-, \varphi_+)$ is a morphism between $\mathbf{A}$ and $\mathbf{B}$, then $\varphi^\perp = (\varphi_+, \varphi_-)$ is a morphism between $\mathbf{B}^\perp$ and $\mathbf{A}^\perp$.

(2) Suppose that $Z \in \mathfrak{d}(\mathbf{B})$ is such that $|Z| = \|\mathbf{B}\|$. Then $\{\varphi_+(z) : z \in Z\}$ is cofinal in A_+. In other words, $\|\mathbf{A}\| \leq |Z|$. □

For two Polish spaces X, Y (i.e. metric, separable with no isolated points) define $\mathsf{BOREL}(X, Y)$ to be the space of all Borel functions from X to Y.

Given relation $\mathbf{A}$ and assuming that both A_- and A_+ are Polish spaces we define families of small sets of reals as:

$$\mathsf{D}(\mathbf{A}) = \{X \subseteq \mathbb{R} : \forall f \in \mathsf{BOREL}(\mathbb{R}, A_+)\ f``(X) \notin \mathfrak{d}(\mathbf{A})\}$$

and

$$\mathsf{B}(\mathbf{A}) = \{X \subseteq \mathbb{R} : \forall f \in \mathsf{BOREL}(\mathbb{R}, A_-)\ f``(X) \notin \mathfrak{b}(\mathbf{A})\}$$

In other words, $\mathsf{D}(\mathbf{A})$ consists of sets of reals whose Borel images are not "dominating" and $\mathsf{B}(\mathbf{A})$ consists of sets whose Borel images are "bounded".

Lemma 2.6. 1. $\mathsf{non}\big(\mathsf{D}(\mathbf{A})\big) = \|\mathbf{A}\|$ *and* $\mathsf{non}\big(\mathsf{B}(\mathbf{A})\big) = \|\mathbf{A}^\perp\|$.
2. *If there exists a Borel morphism from $\mathbf{A}$ to $\mathbf{B}$, then $\mathsf{B}(\mathbf{B}) \subseteq \mathsf{B}(\mathbf{A})$ and $\mathsf{D}(\mathbf{A}) \supseteq \mathsf{D}(\mathbf{B})$.*

Proof. (1) Clearly $\mathsf{non}(\mathsf{D}(\mathbf{A}) \geq \|\mathbf{A}\|$. To show the other inequality notice that there is a Borel mapping from $\mathbb{R}$ onto A_+.

(2) Suppose that $X \notin \mathsf{B}(\mathbf{A})$ and let $f : \mathbb{R} \longrightarrow A_-$ be a Borel map such that $f``(X) \in \mathfrak{b}(\mathbf{A})$. It follows that $\varphi_- \circ f``(X) \in \mathfrak{b}(\mathbf{B})$. Since $\varphi_- \circ f$ is a Borel mapping it follows that $X \notin \mathsf{B}(\mathbf{B})$. □

For cardinals $\kappa = \|\mathbf{A}\|$ and $\lambda = \|\mathbf{B}\|$ the question whether the inequality $\kappa \leq \lambda$ is provable in ZFC leads naturally to the question whether $\mathbf{A} \preceq \mathbf{B}$ and $\mathsf{D}(\mathbf{A}) \subseteq \mathsf{D}(\mathbf{B})$. Even though these questions are more general, in most cases the proof that $\kappa \leq \lambda$ yields $\mathbf{A} \preceq \mathbf{B}$. Moreover, the existence of a Borel morphism witnessing that $\mathbf{A} \preceq \mathbf{B}$ uncovers the combinatorial aspects of these problems.

Historical remarks Tukey embeddings were defined in [53] and further studied in [20]. In context of the orderings considered here see [15], [16] and [28].

The framework used in the definition 2.3 is due to Vojtáš [54]; the particular formulation used here comes from [10].

3. Inequalities provable in ZFC

The notions defined in the previous section are quite general. The focus of this chapter is on the ideal of meager sets ($\mathcal{M}$) and measure zero (null) sets ($\mathcal{N}$) with respect to the standard product measure on μ on ${}^{\omega}2$ or the Lebesgue measure μ on $\mathbb{R}$.

For an ideal $\mathcal{J}$, we identify a Borel mapping $H : \mathbb{R} \longrightarrow \mathcal{J}$ with a Borel set $H \subseteq \mathbb{R} \times \mathbb{R}$ in such a way that

1. $H(x) = (H)_x = \{y : (x, y) \in H\}$.
2. H is a Borel $\mathcal{J}$-set, that is, $(H)_x \in \mathcal{J}$ for all $x \in \mathbb{R}$.

Using this terminology we define the following classes of small sets:

$$\mathsf{COF}(\mathcal{N}) = \mathsf{D}(\mathcal{N}, \mathcal{N}, \subseteq) = \big\{X \subseteq \mathbb{R} : \text{for every Borel } \mathcal{N}\text{-set } H,\ \{(H)_x : x \in X\} \text{ is not a basis of } \mathcal{N}\big\},$$

$$\mathsf{ADD}(\mathcal{N}) = \mathsf{B}(\mathcal{N}, \mathcal{N}, \subseteq) = \big\{X \subseteq \mathbb{R} : \text{for every Borel } \mathcal{N}\text{-set } H,\ \textstyle\bigcup_{x\in X}(H)_x \in \mathcal{N}\big\},$$

$$\mathsf{COV}(\mathcal{N}) = \mathsf{D}(\mathbb{R}, \mathcal{N}, \in) = \big\{X \subseteq \mathbb{R} : \text{for every Borel } \mathcal{N}\text{-set } H,\ \textstyle\bigcup_{x\in X}(H)_x \neq \mathbb{R}\big\},$$

$\mathsf{NON}(\mathcal{N}) = \mathsf{B}(\mathbb{R}, \mathcal{N}, \in) = \big\{X \subseteq \mathbb{R} : \text{every Borel image of } X \text{ is in } \mathcal{N}\big\}$,
$\mathsf{D} = \mathsf{D}({}^{\omega}\omega, {}^{\omega}\omega, \leq^\star)$,
$\mathsf{B} = \mathsf{B}({}^{\omega}\omega, {}^{\omega}\omega, \leq^\star)$.

In the same way we define $\mathsf{ADD}(\mathcal{M})$, $\mathsf{COV}(\mathcal{M})$, etc.

Instead of dealing with all null and meager sets we need to consider only suitably chosen cofinal families.

1. $A \in \mathcal{N}$ if and only if there exists a family of basic open sets $\{C_n : n \in \omega\}$ such that $\sum_{n=0}^{\infty} \mu(C_n) < \infty$ and $A \subseteq \bigcap_{n\in\omega} \bigcup_{m>n} C_m$,
2. $A \in \mathcal{M}$ if and only if there is a family of $\{F_n : n \in \omega\}$ of closed nowhere dense sets such that $A \subseteq \bigcup_{n\in\omega} F_n$.

In particular every null set can be covered by a null set of type G_δ and every meager set can be covered by a meager set of type F_σ.

Definition 3.1. *$\mathbb{C} = \{C^n_m : n, m \in \omega\}$ is a family of clopen subsets of ${}^{\omega}2$ such that $\mu(C^n_m) = 2^{-n}$ for each m.*

Lemma 3.2. $A \in \mathcal{N} \iff \exists f \in {}^{\omega}\omega\ \Big(A \subseteq \bigcap_m \bigcup_{n>m} C^n_{f(n)}\Big)$.

Proof. ($\leftarrow$) Note that the set $\bigcup_{n>m} C^n_{f(n)}$ has measure at most 2^{-m}.

($\rightarrow$) For an open set $U \subseteq {}^{\omega}2$ let

$$\widetilde{U} = \Big\{t \in {}^{<\omega}2 : [t] \subseteq U \ \&\ \forall s \subsetneq t\ \big([s] \not\subseteq U\big)\Big\}.$$

Note that $\widetilde{U}$ is a canonical representation of U as a union of disjoint basic intervals.

Find open sets $\{G_n : n \in \omega\}$ covering A such that $\mu(G_n) \leq 2^{-n}$. Let $\{t_n : n \in \omega\}$ be the lexicographic enumeration of $\bigcup_{n\in\omega} \widetilde{G}_n$. Define for $n \in \omega$,

$$h(n+1) = \min \left\{ k > h(n) : \sum_{j=k}^{\infty} \mu\big([t_j]\big) \leq 2^{-n} \right\},$$

and let

$$D_n = \bigcup_{j=h(n)}^{h(n+1)} [t_j].$$

Let $f \in {}^\omega\omega$ be such that $D_n = C^n_{f(n)}$ for each n. □

Definition 3.3. *Let $\{U_n : n \in \omega\}$ be a basis in ${}^\omega 2$ and let $\mathcal{S} = \{S^n_m : n, m \in \omega\}$ be any family of clopen sets. We say that $\mathcal{S}$ is* good *if*

1. *$S^n_m \cap U_n \neq \emptyset$ for $m \in \omega$,*
2. *For any open dense set $U \subseteq {}^\omega 2$ and $n \in \omega$ there is m such that $S^n_m \subseteq U$.*

Lemma 3.4. *Suppose that the family $\mathcal{S} = \{S^n_m : n, m \in \omega\}$ is good. Then*

$$A \in \mathcal{M} \iff \exists f \in {}^\omega\omega \ \left(A \subseteq {}^\omega 2 \setminus \bigcap_m \bigcup_{n>m} S^n_{f(n)} \right).$$

Proof. ($\leftarrow$) Note that the set $\bigcup_{n>m} S^n_{f(n)}$ is open and dense for every m.

($\rightarrow$) Let $\langle F_n : n \in \omega \rangle$ be an increasing sequence of closed nowhere dense sets covering A. For each n let

$$f(n) = \min\{m : U_n \cap S^n_m \cap F_n = \emptyset\}.$$

It is clear that $\bigcup_n F_n \cap \bigcap_m \bigcup_{n>m} S^n_{f(n)} = \emptyset$. □

Define master sets $N, M \subseteq {}^\omega\omega \times {}^\omega 2$ by

$$N = \bigcap_m \bigcup_{n>m} \bigcup_{f \in {}^\omega\omega} \{f\} \times C^n_{f(n)}$$

and

$$M = ({}^\omega\omega \times {}^\omega 2) \setminus \bigcap_m \bigcup_{n>m} \bigcup_{f \in {}^\omega\omega} \{f\} \times S^n_{f(n)}$$

Note that N is a G_δ set while M is an F_σ set. Moreover, $\{(N)_f : f \in {}^\omega\omega\}$ is cofinal in $\mathcal{N}$ and $\{(M)_f : f \in {}^\omega\omega\}$ is cofinal in $\mathcal{M}$.

The following lemma shows that the representation of meager sets does not depend on the choice of good family:

Lemma 3.5. *Suppose that $\mathcal{S} = \{S^n_m : n, m \in \omega\}$ and $\mathcal{T} = \{T^n_m : n, m \in \omega\}$ are good and M and $\overline{M}$ are associated master sets. Then there are Borel mappings $\varphi_-, \varphi_+ : {}^\omega\omega \longrightarrow {}^\omega\omega$ such that*

$$(M)_f \subseteq (M)_{\varphi_+(g)}, \text{ whenever } (\overline{M})_{\varphi_-(f)} \subseteq (\overline{M})_g.$$

Proof. For $f, g \in {}^\omega\omega$ and $n \in \omega$ define

$$\varphi_-(f)(n) = \min\left\{k : T^n_k \subseteq \bigcup_{m \geq n} S^m_{f(m)}\right\}$$

and

$$\varphi_+(g)(n) = \min\left\{k : S^n_k \subseteq \bigcup_{m \geq n} T^m_{g(m)}\right\}.$$

We leave it to the reader to verify that these mappings have the required properties. $\square$

The following two theorems will be helpful in many subsequent constructions.

Theorem 3.6. *Suppose that $H \subseteq {}^\omega 2 \times {}^\omega 2$ is a Borel set.*

1. *$\{x : (H)_x \in \mathcal{N}\}$ is Borel,*
2. *$\{x : (H)_x \in \mathcal{M}\}$ is Borel,*
3. *If U is open and $(H)_x$ is compact for every x, then $\{x : U \cap (H)_x = \emptyset\}$ is Borel,*
4. *If for every x, $(H)_x$ is "large," where large is either "of positive measure" or "nonmeager", then there exists a Borel function $f : {}^\omega 2 \longrightarrow {}^\omega 2$ such that for every x, $f(x) \in (H)_x$.*

Proof. See [26] 16.A for (1) and (2), 18.B for (4). $\square$

Theorem 3.7. *If X and Y are Polish spaces and $f : X \longrightarrow Y$ is a Borel mapping then there is a dense G_δ set $G \subseteq X$ such that $f{\restriction}G$ is continuous.*

Proof. This is a special case of a theorem of Kuratowski; see [26] 8.I. $\square$

Lemmas 3.2 and 3.4 have their two-dimensional analogs.

Lemma 3.8. *The following conditions are equivalent for a Borel set $H \subseteq {}^\omega 2 \times {}^\omega 2$:*

1. $\forall x\ \big((H)_x \in \mathcal{N}\big)$,
2. *For every $\varepsilon > 0$ there exists a Borel set $B \subseteq {}^\omega 2 \times {}^\omega 2$ such that*
   (a) *$H \subseteq B$,*
   (b) *for every x, $(B)_x$ is an open set of measure $< \varepsilon$.*
3. *There exists a Borel function $x \leadsto f_x$ such that*

$$\forall x\ \big((H)_x \subseteq (N)_{f_x}\big).$$

Proof. (2)$\rightarrow$(3) Let $\{B_n : n \in \omega\}$ be a family of Borel sets such that

1. $H \subseteq \bigcap_n B_n$,
2. For every x, $(B_n)_x$ is an open set of measure $< 2^{-n}$.

Look at the proof of the Lemma 3.2 to see that for each x, $(B)_x = (N)_{f_x}$ and the mapping $x \leadsto f_x$ is Borel.

(3) $\rightarrow$(1) is obvious.

(1)$\rightarrow$(2) By induction on complexity we show that for every $\varepsilon > 0$ and a Borel set $H \subseteq {}^\omega 2 \times {}^\omega 2$ there exists a Borel set $B \supseteq H$ such that for every x, $(B)_x$ is open and $\mu\big((B \setminus H)_x\big) < \varepsilon$. The only nontrivial part is to show that if the theorem holds for sets in Σ^0_α, then it holds for any set $A \in \Pi^0_\alpha$. To see this write $A = \bigcap_n A_n$ where $\langle A_n : n \in \omega\rangle$ is a descending sequence of sets in Σ^0_α. For

each n let B_n be the set obtained from induction hypothesis for A_n and $\varepsilon/2$. Let $K^n = \big\{x : \mu\big((A_n \setminus A)_x\big) < \varepsilon/2\big\}$. Each set K^n is Borel. Now define

$$B = B_0 \cap (K_0 \times {}^\omega 2) \cup \bigcup_{n\in\omega} B_{n+1} \cap \big((K^{n+1} \setminus K^n) \times {}^\omega 2\big).$$

□

Lemma 3.9. *The following conditions are equivalent for a Borel set $H \subseteq {}^\omega 2 \times {}^\omega 2$:*

1. $\forall x \ \big((H)_x \in \mathcal{M}\big)$,
2. *There exists a family of Borel sets $\{G_n : n \in \omega\} \subseteq {}^\omega 2 \times {}^\omega 2$ such that*
   (a) *$(G_n)_x$ is a closed nowhere dense set for all $x \in {}^\omega 2$,*
   (b) *$H \subseteq \bigcup_{n\in\omega} G_n$.*
3. *There exists a Borel function $x \rightsquigarrow f_x$ such that*

$$\forall x \ \big((H)_x \subseteq (M)_{f_x}\big).$$

Proof. (1)→(2) By induction on the complexity we show that for any Borel set $H \subseteq {}^\omega 2 \times {}^\omega 2$ there are Borel sets B and $\{F_n : n \in \omega\}$ such that

1. $(B)_x$ is open for every x,
2. $(F_n)_x$ is closed nowhere dense for every x and n,
3. $H \triangle B \subseteq B \cup \bigcup_n F_n$.

As before the nontrivial part is to show the theorem for the class Π^0_α given that it holds for Σ^0_α. Suppose that $A \in \Sigma^0_\alpha$ and B is the set obtained by applying the inductive hypothesis to A. Let $\langle U_n : n \in \omega\rangle$ be an enumeration of basic sets in ${}^\omega 2$. Define for $n \in \omega$,

$$Z_n = \{x : U_n \cap (B)_x = \emptyset\}.$$

Note that sets Z_n are Borel. Let $B' = \bigcup_n Z_n \times U_n$. The vertical sections of the set $F = {}^\omega 2 \times {}^\omega 2 \setminus (B \cup B')$ are closed nowhere dense and $({}^\omega 2 \setminus A) \triangle B' \subseteq F$, which ends the proof.

(2) → (3) For $x \in {}^\omega 2$ let

$$f_x(n) = \min\Big\{k : \forall i \le n \ \big(S^n_k \cap (G_i)_x = \emptyset\big)\Big\}.$$

□

From these two lemmas it follows that:

Lemma 3.10. *Let $\mathcal{I}$ be $\mathcal{N}$ or $\mathcal{M}$ and let I be the associated master set. Then for $X \subseteq \mathbb{R}$:*

1. $X \in \mathsf{ADD}(\mathcal{I}) \iff \forall F \in \mathsf{BOREL}(\mathbb{R}, {}^\omega\omega)\ \exists f \in {}^\omega\omega\ \forall x \in X\ \big((I)_{F(x)} \subseteq (I)_f\big)$,
2. $X \in \mathsf{COF}(\mathcal{I}) \iff \forall F \in \mathsf{BOREL}(\mathbb{R}, {}^\omega\omega)\ \exists f \in {}^\omega\omega\ \forall x \in X\ \big((I)_f \not\subseteq (I)_{F(x)}\big)$,
3. $X \in \mathsf{COV}(\mathcal{I}) \iff \forall F \in \mathsf{BOREL}(\mathbb{R}, {}^\omega\omega)\ \exists z\ \forall x \in X\ \big(z \notin (I)_{F(x)}\big)$,
4. $X \in \mathsf{NON}(\mathcal{I}) \iff \forall F \in \mathsf{BOREL}(\mathbb{R}, {}^\omega\omega)\ \exists f\ \forall x \in X\ \big(F(x) \in (I)_f\big)$.

The goal of this section is to establish:

Theorem 3.11.

$$\begin{tikzcd}
(\mathbb{R},\mathcal{N},\in) \arrow[r, "\preceq"] & (\mathcal{M},\mathbb{R},\not\ni) \arrow[r, "\preceq"] & (\mathcal{M},\mathcal{M},\subseteq) \arrow[r, "\preceq"] & (\mathcal{N},\mathcal{N},\subseteq) \\
 & ({}^\omega\omega,{}^\omega\omega,\not\geq^\star) \arrow[u, "\preceq"] \arrow[r, "\preceq"] & ({}^\omega\omega,{}^\omega\omega,\leq^\star) \arrow[u, "\preceq"] & \\
(\mathcal{N},\mathcal{N},\not\supseteq) \arrow[uu, "\preceq"] \arrow[r, "\preceq"] & (\mathcal{M},\mathcal{M},\not\supseteq) \arrow[u, "\preceq"] \arrow[r, "\preceq"] & (\mathbb{R},\mathcal{M},\in) \arrow[u, "\preceq"] \arrow[r, "\preceq"] & (\mathcal{N},\mathbb{R},\not\ni) \arrow[uu, "\preceq"]
\end{tikzcd}$$

As a consequence we will get the following two diagrams:

$$\begin{tikzcd}
\mathsf{COV}(\mathcal{N}) \arrow[r, "\subseteq"] & \mathsf{NON}(\mathcal{M}) \arrow[r, "\subseteq"] & \mathsf{COF}(\mathcal{M}) \arrow[r, "\subseteq"] & \mathsf{COF}(\mathcal{N}) \\
 & \mathsf{B} \arrow[u, "\subseteq"] \arrow[r, "\subseteq"] & \mathsf{D} \arrow[u, "\subseteq"] & \\
\mathsf{ADD}(\mathcal{N}) \arrow[uu, "\subseteq"] \arrow[r, "\subseteq"] & \mathsf{ADD}(\mathcal{M}) \arrow[u, "\subseteq"] \arrow[r, "\subseteq"] & \mathsf{COV}(\mathcal{M}) \arrow[u, "\subseteq"] \arrow[r, "\subseteq"] & \mathsf{NON}(\mathcal{N}) \arrow[uu, "\subseteq"]
\end{tikzcd}$$

$$\begin{tikzcd}
\mathsf{cov}(\mathcal{N}) \arrow[r, "\leq"] & \mathsf{non}(\mathcal{M}) \arrow[r, "\leq"] & \mathsf{cof}(\mathcal{M}) \arrow[r, "\leq"] & \mathsf{cof}(\mathcal{N}) \\
 & \mathfrak{b} \arrow[u, "\leq"] \arrow[r, "\leq"] & \mathfrak{d} \arrow[u, "\leq"] & \\
\mathsf{add}(\mathcal{N}) \arrow[uu, "\leq"] \arrow[r, "\leq"] & \mathsf{add}(\mathcal{M}) \arrow[u, "\leq"] \arrow[r, "\leq"] & \mathsf{cov}(\mathcal{M}) \arrow[u, "\leq"] \arrow[r, "\leq"] & \mathsf{non}(\mathcal{N}) \arrow[uu, "\leq"]
\end{tikzcd}$$

The last of these diagrams is called the Cichoń diagram.

It is enough to find the following morphisms:

1. $(\mathbb{R},\mathcal{N},\in) \preceq (\mathcal{M},\mathbb{R},\not\ni)$,
2. $(\mathcal{M},\mathcal{M},\subseteq) \preceq (\mathcal{N},\mathcal{N},\subseteq)$,
3. $(\mathcal{M},\mathcal{M},\not\supseteq) \preceq ({}^\omega\omega,{}^\omega\omega,\not\geq^\star)$,
4. $({}^\omega\omega,{}^\omega\omega,\not\geq^\star) \preceq (\mathcal{M},\mathbb{R},\not\ni)$,
5. $(\mathcal{N},\mathcal{N},\not\supseteq) \preceq (\mathbb{R},\mathcal{N},\in)$.

The remaining morphisms are dual to those listed above. In each case we will find a Borel morphism. Note that thanks to master sets M and N defined earlier, Borel morphisms between these structures can be interpreted as the automorphisms of the index set i.e. ${}^\omega\omega$.

Theorem 3.12. *$\mathcal{M} \preceq \mathcal{N}$; there are two Borel functions $\varphi_-, \varphi_+ : {}^\omega\omega \longrightarrow {}^\omega\omega$ such that*

$$(M)_f \subseteq (M)_{\varphi_+(g)}, \text{ whenever } (N)_{\varphi_-(f)} \subseteq (N)_g.$$

In particular, $\mathsf{ADD}(\mathcal{N}) \subseteq \mathsf{ADD}(\mathcal{M})$ *and* $\mathsf{COF}(\mathcal{M}) \subseteq \mathsf{COF}(\mathcal{N})$, $\mathsf{add}(\mathcal{N}) \leq \mathsf{add}(\mathcal{M})$ *and* $\mathsf{cof}(\mathcal{M}) \leq \mathsf{cof}(\mathcal{N})$.

Proof. Let

$$\mathcal{C} = \left\{ S \in {}^{\omega}({}^{<\omega}[\omega]) : \sum_{n=1}^{\infty} \frac{|S(n)|}{2^n} < \infty \right\}.$$

For $S, S' \in \mathcal{C}$ define $S \subseteq^\star S'$ if for all but finitely many n, $S(n) \subseteq S'(n)$.

Lemma 3.13. $\mathcal{N} \equiv \mathcal{C}$.

Proof. To see that $\mathcal{N} \preceq \mathcal{C}$ define $\varphi_- : {}^{\omega}\omega \longrightarrow \mathcal{C}$ and $\varphi_+ : \mathcal{C} \longrightarrow {}^{\omega}\omega$ such that for $f \in {}^{\omega}\omega$ and $S \in \mathcal{C}$ we have

$$(N)_f \subseteq (N)_{\varphi_+(S)}, \text{ whenever } \varphi_-(f) \subseteq^\star S.$$

For $f \in {}^{\omega}\omega$ put $\varphi_-(f) = h$, where $h(n) = \{f(2n), f(2n+1)\}$. If $S \in \mathcal{C}$ let $\varphi_+(S) = g \in {}^{\omega}\omega$ be such that

$$C^n_{g(n)} = \bigcup_{k \in S(n)} C^{2n}_k \cup \bigcup_{k \in S(n)} C^{2n+1}_k.$$

Verification that these mappings have the required properties is straightforward.

To show that $\mathcal{C} \preceq \mathcal{N}$ we will find Borel functions $\varphi_- : \mathcal{C} \longrightarrow {}^{\omega}\omega$ and $\varphi_+ : {}^{\omega}\omega \longrightarrow \mathcal{C}$ such that for $S \in \mathcal{C}$ and $f \in {}^{\omega}\omega$,

$$S \subseteq^\star \varphi_+(f) \text{ whenever } (N)_{\varphi_-(S)} \subseteq (N)_f.$$

Let $\{G^n_m : n, m \in \omega\}$ be a family of clopen probabilistically independent sets such that $\mu(G^n_m) = 2^{-n}$. For $S \in \mathcal{C}$ define $\varphi_-(S) = f \in {}^{\omega}\omega$ such that

$$\bigcap_{m \in \omega} \bigcup_{n > m} \bigcup_{k \in S(n)} G^n_k \subseteq (N)_f.$$

First consider $H' \subseteq \mathcal{C} \times {}^{\omega}2$ defined by $(H')_S = \bigcap_{m\in\omega} \bigcup_{n>m} \bigcup_{k \in S(n)} G^n_k$ for $S \in \mathcal{C}$. Note that H' is a Borel set and $(H')_S$ has measure zero for every S. Fix a Borel isomorphism $a : \mathcal{C} \longrightarrow {}^{\omega}\omega$ and let $H \subseteq {}^{\omega}\omega \times {}^{\omega}2$ be defined as $(H)_{a(S)} = (H')_S$ for $S \in \mathcal{C}$. Apply 3.8 to find a Borel mapping $x \rightsquigarrow f_x$ such that $(H)_x \subseteq (N)_{f_x}$ and define $\varphi_-(S) = f_{a(S)}$.

To define $\varphi_+ : {}^{\omega}\omega \longrightarrow \mathcal{C}$ we proceed as follows. Find a Borel set $K \subseteq {}^{\omega}\omega \times {}^{\omega}2$ such that

1. $(K)_f$ is a compact set of measure $\geq 1/2$ for all $f \in {}^{\omega}\omega$.
2. $N \cap K = \emptyset$.
3. For any basic open set $U \subseteq {}^{\omega}2$ and $f \in {}^{\omega}\omega$, if $U \cap (K)_f \neq \emptyset$ then $U \cap (K)_f$ has positive measure.

First use Lemma 3.8 to find a set K' satisfying the first two conditions. Let $\langle U_j : j \in \omega \rangle$ be an enumeration of basic open sets in ${}^{\omega}2$. For each j let $Z_j = \{f : \mu(U_j \cap (K')_f) = 0\}$. By Theorem 3.6, the sets Z_j are Borel for each j. Define $K = K' \setminus \bigcup_j (Z_j \times U_j)$.

For $f \in {}^{\omega}\omega, j, n \in \omega$ define

$$S^f_j(n) = \{i \in \omega : (K)_f \cap U_j \neq \emptyset \;\&\; (K)_f \cap U_j \cap G^n_i = \emptyset\}.$$

Note that

$$0 < \mu((K)_f \cap U_j) \leq \prod_n \prod_{i \in S^f_j(n)} \mu({}^{\omega}2 \setminus G^n_i).$$

Thus
$$0<\prod_{n=1}^{\infty}\left(1-\frac{1}{2^n}\right)^{|S^f_j(n)|}.$$
It follows that
$$\sum_{n=1}^{\infty}\frac{|S^f_j(n)|}{2^n}<\infty,$$
so $S^f_j \in \mathcal{C}$ for each j. Moreover, the mapping $f \rightsquigarrow \langle S^f_j : j \in \omega\rangle \in {}^{\omega}\mathcal{C}$ is Borel (by Theorem 3.6(3)). Fix a Borel mapping from ${}^{\omega}\mathcal{C}$ to $\mathcal{C}$ such that $\langle S^f_j : j \in \omega\rangle \rightsquigarrow S^f_\infty$ such that
$$\forall j\ \forall^\infty n\ S^f_j(n) \subseteq S^f_\infty(n).$$
Finally define φ_+ by the formula:
$$\varphi_+(f)(n) = S^f_\infty(n).$$
Suppose that for some $S \in \mathcal{C}$, $(N)_{\varphi_-(S)} \subseteq (N)_f$. It follows that,
$$(K)_f \cap \bigcap_m \bigcup_{n>m} \bigcup_{k \in S(n)} G^n_k = \emptyset.$$
By the Baire category theorem, there is a basic open set U_j and $m_0 \in \omega$ such that $U_j \cap (K)_f \neq \emptyset$ but
$$(K)_f \cap U_j \cap \bigcup_{n>m_0} \bigcup_{k \in S(n)} G^n_k = \emptyset.$$
Therefore
$$\forall^\infty n\ S(2) \subseteq S^f_j(n) \subseteq S^f_\infty(n) = \varphi_+(f)(n),$$
which finishes the proof. □

Lemma 3.14. *$\mathcal{M} \preceq \mathcal{C}$; there are Borel mappings $\varphi_- : {}^{\omega}\omega \longrightarrow \mathcal{C}$ and $\varphi_+ : \mathcal{C} \longrightarrow {}^{\omega}\omega$ such that for any $f \in {}^{\omega}\omega$ and $S \in \mathcal{C}$,*
$$(M)_f \subseteq (M)_{\varphi_+(S)} \text{ whenever } \varphi_-(f) \subseteq^\star S.$$

Proof. We will need the following lemma:

Lemma 3.15. *There exists a good family $\{S^n_m : n, m \in \omega\}$ such that*
$$\forall X \in [\omega]^{\leq 2^n}\ \left(\bigcap_{j \in X} S^n_j \neq \emptyset\right).$$

Proof. Fix $n \in \omega$. Let $\langle C_m : m \in \omega\rangle$ be an enumeration of all clopen sets. For $k \in \omega$ define
$$A_k = \left\{ l > k : C_l \cap \bigcap_{i \in I} C_i \cap U_n \neq \emptyset \text{ whenever } I \subseteq k+1 \text{ and } U_n \cap \bigcap_{i \in I} C_i \neq \emptyset \right\}.$$
Consider family
$$\mathcal{S}_n = \left\{ \bigcup_{i \leq 2^n} C_{m_i} : m_0 \in \omega \text{ and } \ m_{i+1} \in A_{m_i} \text{ for } i \leq 2^n \right\}.$$
We have to check that $\mathcal{S}_n$ satisfies conditions of Definition 3.3.

(1) Let U be a dense open subset of ${}^{\omega}2$. Note that $A_k \cap \{l \in \omega : U_n \cap C_l \subseteq U\} \neq \emptyset$ for every $k \in \omega$, by the density of U.

Now define by induction a sequence $\{m_i : i \leq 2^n\}$ such that $C_{m_i} \subseteq U$ and $m_{i+1} \in A_{m_i}$ for $i < 2^n$. Clearly $U \supseteq \bigcup_{i \leq 2^n} C_{m_i} \in \mathcal{S}_n$.

(2) Suppose that $V_1, V_2, \ldots, V_{2^n} \in \mathcal{S}_n$. For any $j \leq 2^n$, $V_j = \bigcup_{i \leq 2^n} C_{m_i^j}$, where $m_i^j \in A_{m_i^j}$ for $i, j \leq 2^n$. Order the sets V_j in such a way that $m_i^i \leq m_i^j$ for $i \leq j \leq 2^n$.

It is easy to show by induction that $\bigcap_{j \leq 2^n} V_j \supseteq \bigcap_{j \leq 2^n} C_{m_j^j} \neq \emptyset$. □

Let $\mathcal{S} = \bigcup_n \mathcal{S}_n = \{S_m^n : n, m \in \omega\}$. For $f \in {}^\omega\omega$ define $\varphi_-(f) = f \in \mathcal{C}$. For $S \in \mathcal{C}$ let $\varphi_+(S) = f \in {}^\omega\omega$ be such that

$$(M)_f \supseteq {}^\omega 2 \setminus \bigcap_{m \in \omega} \bigcup_{n > m} \bigcap_{i \in S(n)} S_i^n.$$

Since $|S(n)| \leq 2^n$ for all but finitely many n, by Lemma 3.15,

$$\emptyset \neq U_n \cap \bigcap_{i \in S(n)} S_i^n.$$

Now suppose that $\varphi_-(f) \subseteq^\star S$. This assumption means that there exists $n_0 \in \omega$ such that $f(m) \in S(m)$ for $m \geq n_0$. It follows that

$$(M)_{\varphi_+(S)} \supseteq {}^\omega 2 \setminus \bigcap_{m \in \omega} \bigcup_{n > m} \bigcap_{i \in S(n)} S_i^n \supseteq {}^\omega 2 \setminus \bigcap_{m \in \omega} \bigcup_{n > m} S_{f(n)}^n.$$

□

Theorem 3.12 follows immediately; compose the morphisms constructed in Lemma 3.13 and Lemma 3.14. □

Theorem 3.16. $(\mathbb{R}, \mathcal{N}, \in) \preceq (\mathcal{M}, \mathbb{R}, \not\ni)$; *there are Borel functions* $\varphi_-, \varphi_+ : \mathbb{R} \longrightarrow {}^\omega\omega$ *such that for* $x, y \in \mathbb{R}$,

$$x \in (N)_{\varphi_+(y)} \text{ whenever } y \notin (M)_{\varphi_-(x)}.$$

In particular, $\mathsf{COV}(\mathcal{N}) \subseteq \mathsf{NON}(\mathcal{M})$ *and* $\mathsf{COV}(\mathcal{M}) \subseteq \mathsf{NON}(\mathcal{N})$, $\mathsf{cov}(\mathcal{N}) \leq \mathsf{non}(\mathcal{M})$ *and* $\mathsf{cov}(\mathcal{M}) \leq \mathsf{non}(\mathcal{N})$.

Proof. Let B be a G_δ null set whose complement is meager. Use Theorem 3.6(3) and Theorem 3.7 to find Borel functions $\varphi_-, \varphi_+ : \mathbb{R} \longrightarrow {}^\omega\omega$ such that

$$\forall x \ \left(B + x \subseteq (N)_{\varphi_-(x)}\right) \text{ and } \forall y \ \left({}^\omega 2 \setminus (B + y) \subseteq (M)_{\varphi_+(y)}\right).$$

If $y \notin (N)_{\varphi_-(x)}$ then $y \notin B + x$. It follows that $x \in {}^\omega 2 \setminus (B + y) \subseteq (M)_{\varphi_+(y)}$. □

Theorem 3.17. $(\mathcal{M}, \mathcal{M}, \not\supseteq) \preceq ({}^\omega\omega, {}^\omega\omega, {}^\star\not\geq)$; *there are Borel mappings* $\varphi_-, \varphi_+ : {}^\omega\omega \longrightarrow {}^\omega\omega$ *such that for* $f, g \in {}^\omega\omega$

$$(M)_f \not\supseteq (M)_{\varphi_+(g)}, \text{ whenever } \varphi_-(f)^\star \not\geq g.$$

In particular, $\mathsf{D} \subseteq \mathsf{COF}(\mathcal{M})$ *and* $\mathsf{ADD}(\mathcal{M}) \subseteq \mathsf{B}$, $\mathfrak{d} \leq \mathsf{cof}(\mathcal{M})$ *and* $\mathsf{add}(\mathcal{M}) \leq \mathfrak{b}$.

Proof. Let $\mathcal{S}_n$ be the family of clopen sets C such that there exists $k > n$ and $s \in {}^{[n,k)}2$ such that

$$C = \{x \in {}^\omega 2 : x{\restriction}[n, k) = s\}.$$

Note that the family $\mathcal{S} = \bigcup_n \mathcal{S}_n$ is good (given the appropriate choice of the sequence $\{U_n : n \in \omega\}$).

For $f \in {}^\omega\omega$ let $\varphi_-(f)(n) = k$ if and only if $\mathsf{dom}(S_{f(n)}^n) = [n, k)$.

For a strictly increasing function $g \in {}^\omega\omega$ define $\varphi_+(f) = h \in {}^\omega\omega$ such that

$$(M)_h = \{x \in {}^\omega 2 : \forall^\infty n\ \exists i \in \big[n, f(n)\big)\ \big(x(i) \neq 0\big)\}.$$

Note that the image of ${}^\omega\omega$ under φ_+ is rather small, $\varphi_+``({}^\omega\omega)$ is not even cofinal in $\mathcal{M}$.

To finish the proof it is enough to show that if $\varphi_-(f)(n) < g(n)$ for infinitely many n, then

$$\big\{x \in {}^\omega 2 : \forall^\infty n\ \exists i \in \big[n, g(n)\big)\ x(i) \neq 0\big\} \not\subseteq \Big\{x \in {}^\omega 2 : \forall^\infty n\ x{\restriction}\big[n, \varphi_-(f)(n)\big) \neq S^n_{f(n)}\Big\}.$$

Find a sequence $\{n_k : k \in \omega\}$ such that for all k,

$$n_k < \varphi_-(f)(n_k) < g(n_k) < n_{k+1}.$$

Construct a real z such that $z{\restriction}\big[n_k, \varphi_-(f)(n_k)\big) = S^{n_k}_{f(n_k)}$. Thus $z \in (M)_f$ but $z{\restriction}\big[n, g(n)\big) \not\equiv 0$ for all n, so $z \in \{x \in {}^\omega 2 : \forall^\infty n\ \exists i \in \big[n, f(n)\big)\ (x(i) \neq 0)\}$. $\square$

Theorem 3.18. $({}^\omega\omega, {}^\omega\omega, \not\geq) \preceq (\mathbb{R}, \mathcal{M}, \not\ni)$*; there are mappings* $\varphi_- : {}^\omega\omega \longrightarrow {}^\omega\omega$ *and* $\varphi_+ : \mathbb{R} \longrightarrow {}^\omega\omega$ *such that for* $f \in {}^\omega\omega$ *and* $y \in \mathbb{R}$,

$$f \not\geq^\star \varphi_+(y) \text{ whenever } y \notin (M)_{\varphi_-(f)}.$$

In particular, $\mathsf{COV}(\mathcal{M}) \subseteq \mathsf{D}$ *and* $\mathsf{B} \subseteq \mathsf{NON}(\mathcal{M})$*,* $\mathsf{cov}(\mathcal{M}) \leq \mathfrak{d}$ *and* $\mathfrak{b} \leq \mathsf{non}(\mathcal{M})$*.*

Proof. Identify $\mathbb{R} \setminus \mathbb{Q}$ with ${}^\omega\omega$ and define $\varphi_-(f) = h$ such that

$$(M)_h = \big\{z \in {}^\omega\omega : \forall^\infty n\ \big(z(n) \leq f(n)\big)\big\}$$

and $\varphi_+(y) = y$. $\square$

Theorem 3.19. $(\mathcal{N}, \mathcal{N}, \not\supseteq) \preceq (\mathcal{N}, \mathbb{R}, \in)$*; there are Borel functions* $\varphi_- : {}^\omega\omega \longrightarrow \mathbb{R}$ *and* $\varphi_+ : {}^\omega\omega \longrightarrow {}^\omega\omega$ *such that for* $f, g \in {}^\omega\omega$,

$$(N)_f \not\supseteq (N)_{\varphi_+(g)}, \text{ whenever } \varphi_-(f) \in (N)_g.$$

Proof. Let $\varphi_- : {}^\omega\omega \longrightarrow \mathbb{R}$ be any Borel function such that for $f \in {}^\omega\omega$, $\varphi_-(f) \notin (N)_f$ (see Theorem 3.6(4)) and let $\varphi_+(g) = g$ for $g \in {}^\omega\omega$. Verification that both functions have the required properties is straightforward. $\square$

We conclude this section with some remarks concerning Luzin sets.

Definition 3.20. *Given* $\mathbf{A} = (A, A_-, A_+)$ *and two cardinals* $\kappa \leq \lambda$ *we call a set* $X \subseteq A_-$ *a Luzin set if* $|X| \geq \lambda$ *and for every* $Y \subseteq X$, $|Y| = \kappa$, $Y \in \mathfrak{b}(\mathbf{A})$*.*

When $\mathbf{A} = (\mathbb{R}, \mathcal{M}, \in)$, $\kappa = \aleph_1$ and $\lambda = 2^{\aleph_0}$ then we get the original Luzin set. The set given by $(\mathbb{R}, \mathcal{N}, \in)$, $\kappa = \aleph_1$ and $\lambda = 2^{\aleph_0}$ is usually called Sierpinski set.

Lemma 3.21. *Suppose that* X *is a Luzin set determined by* $\mathbf{A}$ *and* $\kappa \leq \lambda$*. Then* $\|\mathbf{A}\| \geq \lambda$ *and* $\|\mathbf{A}^\perp\| \leq \kappa$*.*

Proof. Since every set $Y \subseteq X$, $|Y| = \kappa$ belongs to $\mathfrak{b}(\mathbf{A}) = \mathfrak{d}(\mathbf{A}^\perp)$, we get the second inequality.

For the first inequality note that if $y \in A_+$ then $\{x \in X \cap A_- : A(x, y)\}$ has size $< \kappa \leq |X|$. Thus any family that dominates X has to have a size at least $|X| \geq \lambda$. $\square$

Morphisms preserve Luzin sets.

Lemma 3.22. *Suppose that* $\mathbf{A} \leq \mathbf{B}$ *and* X *is a* (κ, λ) *Luzin set in* $\mathbf{A}$. *Then* $\varphi_{-}``(X)$ *is a* κ, λ *Luzin set in* $\mathbf{B}$.

Proof. Clearly every subset of size κ of $\varphi_{-}``(X)$ is unbounded. Moreover, for every $B \in B_{-}$, $\varphi^{-1}(B) \cap X$ has size $< \kappa$. Thus $|\varphi_{-}``(X)| \geq \lambda$. $\square$

Historical remarks Families of small sets as defined here appeared in various contexts. Recław [37] suggested considering small sets rather than cardinal characteristics.

Many people contributed to the proof of the Theorem 3.11. In the last diagram:

- Rothberger [40] showed that $\mathsf{cov}(\mathcal{M}) \leq \mathsf{non}(\mathcal{N})$ and $\mathsf{cov}(\mathcal{N}) \leq \mathsf{non}(\mathcal{M})$.
- Miller [29] and Truss [52] showed that $\mathsf{add}(\mathcal{M}) = \min\{\mathfrak{b}, \mathsf{cov}(\mathcal{M})\}$ and Fremlin showed that $\mathsf{cof}(\mathcal{M}) = \max\{\mathfrak{d}, \mathsf{non}(\mathcal{M})\}$.
- Bartoszynski [2] and Raisonnier and Stern in their [35] showed that $\mathsf{add}(\mathcal{N}) \leq \mathsf{add}(\mathcal{M})$ and $\mathsf{cof}(\mathcal{M}) \leq \mathsf{cof}(\mathcal{N})$. Different proofs of these inequalities have been found — forcing proof by Judah and Repický [22] and a very general combinatorial argument, Theorem 4.23 of this paper.

Fremlin [14] first realized that Tukey embeddings are responsible for the inequalities in the Cichoń diagram. Pawlikowski [32] proved Lemma 3.14, which was the crucial step in the proof of $\mathcal{M} \preceq \mathcal{N}$.

The first diagram of Theorem 3.11:

- Vojtáš [54] proved it with arbitrary morphisms,
- Recław [37] proved a version with Borel morphisms (which gives the second diagram),
- Pawlikowski and Recław in their [34] proved the existence of continuous morphisms.

Lemma 3.22 was proved in [13].

4. Combinatorial characterizations

This section is devoted to the combinatorics associated with the cardinal invariants of the Cichoń diagram. We will find the combinatorial equivalents of most of the invariants as well as characterize membership in the corresponding classes of small sets. We conclude the section with a characterization of the ideal $(\mathcal{N}, \subseteq)$ as maximal in the sense of Tukey connections among a large class of partial orderings.

Theorem 4.1. *The following conditions are equivalent:*

1. $X \in \mathsf{COV}(\mathcal{M})$,
2. *for every Borel function* $x \rightsquigarrow f^x \in {}^\omega\omega$ *there exists a function* $g \in {}^\omega\omega$ *such that*
$$\forall x \in X \ \exists^\infty n \ \big(f^x(n) = g(n)\big).$$

Proof. (1) $\rightarrow$ (2). Suppose that $x \rightsquigarrow f^x \in {}^\omega\omega$ is a Borel mapping. Let $H = \{\langle x, h\rangle \in {}^\omega 2 \times {}^\omega\omega : \forall^\infty n \ \big(h(n) \neq f^x(n)\big)\}$. Clearly H is a Borel set with all $(H)_x$ meager and if $g \notin \bigcup_{x \in X} (H)_x$ then g has required properties.

(2) $\rightarrow$ (1). We will need several lemmas. To avoid repetitions let us define:

Definition 4.2. *Suppose that* $X \subseteq {}^\omega 2$. X *is nice if for every Borel function* $x \rightsquigarrow f^x \in {}^\omega\omega$ *there exists a function* $g \in {}^\omega\omega$ *such that*
$$\forall x \in X \ \exists^\infty n \ \big(f^x(n) = g(n)\big).$$

Lemma 4.3. *Suppose that X is nice. Then for every Borel function $x \rightsquigarrow \langle Y^x, f^x\rangle \in {}^\omega[\omega] \times {}^\omega\omega$ there exists $g \in {}^\omega\omega$ such that*

$$\forall x \in X\ \exists^\infty n \in Y^x\ (f^x(n) = g(n)).$$

Proof. Suppose that a Borel mapping $x \rightsquigarrow \langle Y^x, f^x\rangle$ is given. Let y^x_n denote the n-th element of Y^x for $x \in {}^\omega 2$. For every $x \in {}^\omega 2$ define a function h^x as follows:

$$h^x(n) = f^x \restriction \{y^x_0,\ y^x_1, \dots, y^x_n\} \text{ for } n \in \omega.$$

Since the mapping $x \rightsquigarrow h^x$ is Borel and functions h^x can be coded as elements of ${}^\omega\omega$ there is a function h such that

$$\forall x \in X\ \exists^\infty n\ \big(h^x(n) = h(n)\big).$$

Without loss of generality we can assume that $h(n)$ is a function from an $n+1$-element subset of ω into ω.

Define $g \in {}^\omega\omega$ in the following way. Recursively choose

$$z_n \in \mathsf{dom}\big(h(n)\big) \setminus \{z_0, z_1, \dots, z_{n-1}\} \text{ for } n \in \omega.$$

Then let g be any function such that $g(z_n) = h(n)(z_n)$ for $n \in \omega$.

We show that the function g has the required properties. Suppose that $x \in X$. Notice that the equality $h^x(n) = h(n)$ implies that

$$f^x(z_n) = g(z_n) \text{ and } z_n \in Y^x.$$

That finishes the proof since $h^x(n) = h(n)$ for infinitely many $n \in \omega$. □

Lemma 4.4. *Suppose that X is nice. Then for every Borel mapping $x \rightsquigarrow f^x \in {}^\omega\omega$ there exists an increasing sequence $\langle n_k : k \in \omega\rangle$ such that*

$$\forall x \in X\ \exists^\infty k\ \big(f^x(n_k) < n_{k+1}\big).$$

Proof. Suppose that the lemma is not true and let $x \rightsquigarrow f^x$ be a counterexample. Without loss of generality we can assume that f^x is increasing for all $x \in X$. To get a contradiction we will define a Borel mapping $x \rightsquigarrow g^x \in {}^\omega\omega$ such that $\{g^x : x \in X\}$ is a dominating family. That will contradict the assumption that X is nice.

Define

$$g^x(n) = \max\{\underbrace{f^x \circ f^x \circ \cdots \circ f^x}_{j+1 \text{ times}}(i) : i, j \le n\} \quad \text{for } n \in \omega.$$

Suppose that $g \in {}^\omega\omega$ is an increasing function. By the assumption there exist $x \in X$ and k_0 such that

$$\forall k \ge k_0\ \big(f^x\big(g(k)\big) \ge g(k+1)\big).$$

In particular,

$$\forall k \ge g(k_0)\ \big(g(k) \le g^x(k)\big)$$

which finishes the proof. □

We now return to the proof that (2) implies (1) for 4.1. Let $x \rightsquigarrow f_x \in {}^\omega\omega$ be a Borel mapping. We want to show that $\bigcup_{x \in X}(M)_{f_x} \ne {}^\omega 2$.

Without loss of generality we can assume that M is the set built using the family from the proof of Lemma 3.17. For each x let $g_x \in {}^\omega\omega$ and $\{s^x_n : n \in \omega\}$ be such that $S^n_{f_x(n)} = \{x \in {}^\omega 2 : x \restriction \big[n, g_x(n)\big) = s^x_n\}$.

By Lemma 4.4, there exists a sequence $\langle n_k : k \in \omega\rangle$ such that

1. $n_{k+1} > \sum_{i=0}^k n_i$, for all k,

2. $\forall x \in X\ \exists^\infty n\ \big(g_x(n_k) < n_{k+1}\big)$.

For $x \in X$ let $Z^x = \{k : g_x(n_k) < n_{k+1}\}$. By Lemma 4.3, there exists a sequence $\langle s_k : k \in \omega\rangle$ such that

$$\forall x \in X\ \exists^\infty k \in Z^x\ \big(s^x_{n_k} = s_k\big).$$

Without loss of generality we can assume that $s_k : [n_k, m_k) \longrightarrow 2$, where $m_k < n_{k+1}$. Choose $z \in {}^\omega 2$ such that $s_k \subseteq z$ for all k. It follows that $z \notin (M)_{f_x}$ for every $x \in X$. □

As a corollary we have:

Theorem 4.5. *The following are equivalent:*

1. $\mathsf{cov}(\mathcal{M}) > \kappa$,
2. $\forall F \subseteq [{}^\omega\omega]^\kappa\ \exists g \in {}^\omega\omega\ \forall f \in F\ \exists^\infty n\ \big(f(n) = g(n)\big)$.

The above proof can be dualized to give:

Theorem 4.6. *The following conditions are equivalent:*

1. $X \times X \in \mathsf{NON}(\mathcal{M})$,
2. *for every Borel function $x \rightsquigarrow f^x \in {}^\omega\omega$ there exists a function $g \in {}^\omega\omega$ such that*
$$\forall x \in X\ \forall^\infty n\ f^x(n) \neq g(n).$$

We only explain why we have $X \times X$ in (1) rather than X. If we analyze the proof of Theorem 4.1, we see that in order to produce real z such that $z \notin \bigcup_{x\in X}(M)_{f_x}$ we had to diagonalize (find an infinitely often equal real) twice.

Similar situation arises here; each element of X produces two functions, and a real that avoids a given meager set is constructed from two such functions, each coming from a different point of X.

As a corollary we get:

Theorem 4.7. $\mathsf{non}(\mathcal{M})$ *is the size of the smallest family $F \subseteq {}^\omega\omega$ such that*

$$\forall g \in {}^\omega\omega\ \exists f \in F\ \exists^\infty n\ \big(f(n) = g(n)\big).$$

Theorem 4.8. $\mathsf{ADD}(\mathcal{M}) = \mathsf{B} \cap \mathsf{COV}(\mathcal{M})$. *In particular,* $\mathsf{add}(\mathcal{M}) = \min\{\mathfrak{b}, \mathsf{cov}(\mathcal{M})\}$.

Proof. The inclusion $\subseteq$ follows immediately from Theorem 3.11.

Suppose that $X \in \mathsf{B} \cap \mathsf{COV}(\mathcal{M})$. Let $x \rightsquigarrow f_x \in {}^\omega\omega$ be a Borel mapping. Since $X \in \mathsf{COV}(\mathcal{M})$ there is a real z such that $z \notin \bigcup_{x\in X}(M)_{f_x}$. For $x \in X$ define for $n \in \omega$,

$$g_x(n) = \min\left\{ l : \forall t \in {}^n 2\ \left(\big[t^\frown z{\restriction}[n,l)\big] \subseteq \bigcup_{m>n} S^m_{f_x(m)} \right) \right\}.$$

The mapping $x \rightsquigarrow g_x$ is also Borel. Since $X \in \mathsf{B}$, it follows that there is an increasing function $h \in {}^\omega\omega$ such that

$$\forall x \in X\ \forall^\infty n\ \big(g_x(n) \leq h(n)\big).$$

Consider the set

$$G = \bigcap_n \bigcup_{m>n} \bigcup \left\{ \Big[t^\frown z{\restriction}\big[m, h(m)\big)\Big] : t \in {}^m 2 \right\}.$$

Clearly G is a dense G_δ set. Moreover, for every $x \in X$ there is n such that

$$\bigcup_{m>n} \bigcup \left\{ \left[t^\frown z{\restriction}[m, h(m)) \right] : t \in {}^m 2 \right\} \subseteq \bigcup_{m>n} S^m_{f_x(m)}.$$

It follows that

$$\bigcup_{x \in X} (M)_{f_x} \subseteq {}^\omega 2 \setminus G,$$

which finishes the proof. $\square$

From the Theorem 3.11 it follows that $\mathsf{D} \cup \mathsf{NON}(\mathcal{M}) \subseteq \mathsf{COF}(\mathcal{M})$. The other inclusion does not hold. We only have the following result dual to Theorem 4.8.

Theorem 4.9. *If $X \notin \mathsf{D}$ and $Y \notin \mathsf{NON}(\mathcal{M})$ then $X \times Y \notin \mathsf{COF}(\mathcal{M})$. In particular,* $\mathsf{cof}(\mathcal{M}) = \max\{\mathsf{non}(\mathcal{M}), \mathfrak{d}\}$.

Definition 4.10. *Let $\mathbb{R}_+ = \{x \in \mathbb{R} : x \geq 0\}$ and define*

$$\ell^1 = \left\{ f \in {}^\omega\mathbb{R}_+ : \sum_{n=1}^\infty f(n) < \infty \right\}.$$

For $f, g \in \ell^1$, $f \leq^\star g$ if $f(n) \leq g(n)$ holds for all but finitely many n.

Theorem 4.11. *The following are equivalent:*

1. $X \in \mathsf{ADD}(\mathcal{N})$,
2. *for every Borel function $x \rightsquigarrow S^x \in \mathcal{C}$ there exists $S \in \mathcal{C}$ such that*
$$\forall x \in X\ \forall^\infty n\ \big(S^x(n) \subseteq S(n)\big).$$
3. *for every Borel function $x \rightsquigarrow f^x \in \ell^1$ there exists a function $f \in \ell^1$ such that*
$$\forall x \in X\ \forall^\infty n\ \big(f^x(n) \leq f(n)\big).$$

In particular, the following conditions are equivalent:

a. $\mathsf{add}(\mathcal{N}) > \kappa$,
b. *for every family $F \subseteq {}^\omega\omega$ of size κ there exists $S \in \mathcal{C}$ such that*
$$\forall f \in F\ \forall^\infty n\ \big(f(n) \in S(n)\big),$$
c. *for every family $F \subseteq \ell^1$ of size κ there exists $g \in \ell^1$ such that*
$$\forall f \in F\ \forall^\infty n\ \big(f(n) \leq g(n)\big).$$

Proof. We will establish the equivalence of (1) and (2). Suppose that $X \in \mathsf{ADD}(\mathcal{N})$ and $x \rightsquigarrow S_x$ is a Borel mapping. Consider the morphism (φ_-, φ_+) witnessing that $\mathcal{C} \preceq \mathcal{N}$. Let f be such that $\bigcup_{x \in X} (N)_{\varphi_-(S_x)} \subseteq (N)_f$. Then $\varphi_+(f) \in \mathcal{C}$ is the object we are looking for.

Suppose that $X \notin \mathsf{ADD}(\mathcal{N})$. Let $F : X \longrightarrow {}^\omega\omega$ be a Borel mapping such that $\bigcup_{x \in X} (N)_{F(x)} \not\subseteq (N)_f$ for $f \in {}^\omega\omega$. Consider the morphism (φ_-, φ_+) witnessing that $\mathcal{N} \preceq \mathcal{C}$. It follows that there is no $S \in \mathcal{C}$ such that

$$\forall x \in X\ \forall^\infty n\ \big(\varphi_-\big(F(x)\big)(n) \subseteq^\star S(n)\big).$$

Equivalence of (2) and (3) follows from:

Lemma 4.12. $\mathcal{C} \equiv \ell^1$.

Proof. To show that $\ell^1 \preceq \mathcal{C}$ define $\varphi_- : \ell^1 \longrightarrow \mathcal{C}$ as

$$\varphi_-(f)(n) = \left\{k : 2^{-n} > f(k) \geq 2^{-n-1}\right\}.$$

Similarly, define $\varphi_+ : \mathcal{C} \longrightarrow \ell^1$ is defined by: $\varphi_+(S)(n) = \max\{2^{-k} : n \in S(k)\}$. It is easy to see that these mappings have the required properties.

To show that $\mathcal{C} \preceq \ell^1$ identify $\omega \times \omega$ with ω via functions $L, K \in {}^\omega\omega$. For $S \in \mathcal{C}$ let

$$\varphi_-(S)(n) = \begin{cases} 2^{-n} & \text{if } K(n) \in S\big(L(n)\big) \\ 0 & \text{otherwise} \end{cases}.$$

For $f \in \ell^1$ let

$$\varphi_+(f)(n) = \left\{k : \frac{1}{2^{n-1}} > f(k) \geq \frac{1}{2^n}\right\}.$$

□

The second part of 4.11 follows readily from the first. □

The dual version yields:

Theorem 4.13. *The following are equivalent:*

1. $X \in \mathsf{COF}(\mathcal{N})$,
2. *for every Borel function $x \rightsquigarrow S^x \in \mathcal{C}$ there exists $S \in \mathcal{C}$ such that*
$$\forall x \in X\ \exists^\infty n\ \big(S(n) \not\subseteq S^x(n)\big).$$
3. *for every Borel function $x \rightsquigarrow f^x \in \ell^1$ there exists a function $f \in \ell^1$ such that*
$$\forall x \in X\ \exists^\infty n\ \big(f^x(n) \leq f(n)\big).$$

In particular, the following are equivalent:

a. $\mathsf{cof}(\mathcal{N}) < \kappa$,
b. *for every family $F \subseteq {}^\omega\omega$ of size κ there exists $S \in \mathcal{C}$ such that*
$$\forall f \in F\ \exists^\infty n\ \big(f(n) \notin S(n)\big),$$
c. *for every family $F \subseteq \ell^1$ of size κ there exists $g \in \ell^1$ such that*
$$\forall f \in F\ \exists^\infty n\ \big(f(n) \leq g(n)\big).$$

Additivity of measure, $\mathsf{add}(\mathcal{N})$, has a special place among cardinal invariants of the continuum as being provably smaller than a large number of them. It has been conjectured (wrongly in [1]) that this is because additivity of measure is equivalent to the Martin Axiom for a large class of forcing notions (Suslin ccc). Only very recently this phenomenon has been explained as being directly related to the combinatorial complexity of the measure ideal.

Definition 4.14. *We say that an ideal $\mathcal{J} \subseteq \mathbf{P}(\omega)$ is a p-ideal if for every family $\{X_n : n \in \omega\} \subseteq \mathcal{J}$ there is $X \in \mathcal{J}$ such that $X_n \subseteq^\star X$ for $n \in \omega$.*

Define

$$\mathsf{add}^\star(\mathcal{J}) = \min\{|\mathcal{A}| : \mathcal{A} \subseteq \mathcal{J}\ \&\ \neg\exists Y \in \mathcal{J}\ \forall X \in \mathcal{A}\ X \subseteq^\star Y\}.$$

It is easy to see that the $\mathsf{cof}^\star$ defined analogously is equivalent to the true cofinality.

Many ideals of Borel subsets of $\mathbb{R}$ are Tukey equivalent to analytic (Σ^1_1) ideals of subsets of ω. For example:

- Ideal of null sets $\mathcal{N}$. Work with $\mathcal{C}$ instead of $\mathcal{N}$. For $S \in \mathcal{C}$ let $A_S = \{(n,k) \in \omega \times \omega : k \in S(n)\}$. The family $\{A_S : S \in \mathcal{C}\}$ generates an analytic p-ideal on $\omega \times \omega \simeq \omega$.
- $({}^\omega\omega, \leq^\star)$. This is very similar. For $f \in {}^\omega\omega$ let $A_f = \{(n,k) : k \leq f(n)\}$.
- The ideal of meager sets $\mathcal{M}$. Let $\{C_n : n \in \omega\}$ be an enumeration of open basic subsets of ω. Consider the ideal generated by sets $X \subseteq \omega$ such that for every k, $\bigcup_{n \notin X \cup k} C_n$ is open dense. This is an analytic p-ideal on ω which is equvalent to $\mathcal{M}$.
- ℓ^1 is Tukey equivalent to the p-ideal of summable sets

$$\left\{X \subseteq \omega : \sum_{n \in X} \frac{1}{n} < \infty\right\}.$$

Moreover, in all these cases the additivity of the ideal is equal to the *additivity of the associated ideal on ω. For example, $\mathsf{add}(\mathcal{N}) = \mathsf{add}^\star(\ell^1) = \mathsf{add}^\star(\mathcal{C})$, etc. In the remainder of this section we will show for a (nontrivial) analytic p-ideal $\mathcal{J}$ on ω we have

$${}^\omega\omega \preceq \mathcal{J} \preceq \mathcal{N}.$$

In particular, by the above remarks, $\mathfrak{b} \geq \mathsf{add}(\mathcal{M}) \geq \mathsf{add}(\mathcal{N})$.

We need a few general facts about analytic p-ideals. To simplify the notation let us identify ${}^\omega 2$ with $\mathbf{P}(\omega)$ via characteristic functions.

Let $\mathsf{K}({}^\omega 2)$ be the collection of compact subsets of ${}^\omega 2$ with Hausdorff metric d_H defined as follows. For two nonempty compact sets $K, L \subseteq {}^\omega 2$ let

$$d_H(K,L) = \max\big(\rho(K,L), \rho(L,K)\big),$$

where $\rho(K,L) = \max_{x \in L} d(x,K)$ (d is the usual metric in ${}^\omega 2$).

Let $\mathsf{M} \subseteq \mathsf{K}({}^\omega 2)$ be the collection of compact subsets of ${}^\omega 2$ which are downward closed. We will use the following well known facts:

Lemma 4.15. 1. $\mathsf{K}({}^\omega 2)$ *is a compact Polish space,*
2. M *is a closed subspace of* $\mathsf{K}({}^\omega 2)$.

Proof. See [26] 4.F. □

Let

$$\mathsf{F} = \{K \in \mathsf{M} : \forall X \in \mathcal{J}\ \exists n\ (X \setminus n \in K)\}.$$

It is clear that F is a filter.

Lemma 4.16. *Suppose that $H \subseteq \mathsf{F}$ is a closed set. There exists a relatively clopen set $U \subseteq H$ such that*

$$\bigcap_{K \in U} K \in \mathsf{F}.$$

In particular, $H = \bigcup_{n \in \omega} H_n$, where for each n, $\bigcap_{K \in H_n} K \in \mathsf{F}$.

Proof. Let $\langle U_n : n \in \omega \rangle$ be an enumeration of clopen subsets of $\mathsf{K}({}^\omega 2)$.

For $X \in \mathcal{J}$ and n define $H_n(X) = \{K \in H : X \setminus n \in K\}$. The sets $H_n(X)$ are closed and $H = \bigcup_{n \in \omega} H_n(X)$ for every $X \in \mathcal{J}$. By the Baire Category theorem for each X there is a pair $\big(n(X), m(X)\big) \in \omega \times \omega$ such that

$$H_{n(X)}(X) \cap U_{m(X)} = H \cap U_{m(X)}.$$

Since $\mathcal{J}$ is a p-ideal, we can find (n,m) such that

$$\{X : n(X) = n \ \&\ m(X) = m\} \text{ is cofinal in } \mathcal{J}.$$

It follows that $\bigcap_{K\in U_m\cap H} K \in \mathsf{F}$. That finishes the proof of the first part.

To prove the second part let $H_1 = H \cap U_m$. Next, apply the above construction to $H \setminus H_1$ (which is closed) to get H_2, etc. □

Lemma 4.17. F *is* F_σ *in* M.

Proof. Consider $\mathsf{G} = \mathsf{M} \setminus \mathsf{F}$. Note that for $K \in \mathsf{M}$ we have

$$K \in \mathsf{G} \iff \exists X \in \mathcal{J}\ \forall n\ (X \setminus n \notin K).$$

It follows that G is an analytic ideal. Moreover, G is a σ-ideal; if $\{K_n : n \in \omega\} \subseteq \mathsf{G}$ and $K \subseteq \bigcup_n K_n$ then $K \in \mathsf{G}$. To see this let X_n witness that $K_n \in \mathsf{G}$. Find $X \in \mathcal{J}$ such that $X_n \subseteq^\star X$ for all n. Clearly, $X \setminus n \notin K$ for all n. Now the lemma follows immediately from the following:

Theorem 4.18. *Let $\mathcal{I}$ be an analytic σ-ideal of compact sets in a compact metrizable space E. Then $\mathcal{I}$ is actually G_δ.*

Proof. See [12], [36] or [25]. □

□

Lemma 4.19. F *is countably generated.*

Proof. Using Lemma 4.17 represent $\mathsf{F} = \bigcup_n H_n$, where each H_n is closed. Apply Lemma 4.16 to write for $n \in \omega$, $H_n = \bigcup_{m\in\omega} H^n_m$, where $G^n_m = \bigcap_{K\in H^n_m} K \in \mathsf{F}$. It is clear that $\{G^n_m : n, m \in \omega\}$ generates F. □

Let $\langle G_n : n \in \omega\rangle$ be a descending sequence generating F. The following lemma gives a simple ($F_{\sigma\delta}$) description of $\mathcal{J}$ in terms of $\langle G_n : n \in \omega\rangle$.

Lemma 4.20. $X \in \mathcal{J} \iff \forall n\ \exists m\ (X \setminus m \in G_n)$.

Proof. Implication ($\rightarrow$) is obvious.

($\leftarrow$) We will use the following result.

Theorem 4.21. *Suppose that $\mathcal{I} \subseteq \mathbf{P}(\omega)$ is an ideal containing all finite sets. The following conditions are equivalent:*

1. *$\mathcal{I}$ has the Baire property,*
2. *$\mathcal{I}$ is meager,*
3. *there exists a partition $\{I_n : n \in \omega\}$ of ω into disjoint intervals such that*

$$\forall X \in \mathcal{I}\ \forall^\infty n\ (I_n \not\subseteq X).$$

Proof. See [48] or [8]. □

Suppose that $X \notin \mathcal{J}$. The ideal $\mathcal{J}{\restriction}X = \{Y \cap X : Y \in \mathcal{J}\} \subseteq \mathbf{P}(X)$ is analytic and hence has the Baire property. By Theorem 4.21(3) there exists a partition $\{I_n : n \in \omega\}$ of X into finite sets such that

$$\forall Z \in \mathcal{J}\ \forall^\infty n\ (I_n \not\subseteq Z).$$

Choose $n_0 \in \omega$ such that the set

$$K = \{Y : \forall n > n_0\ (I_n \not\subseteq Y)\} \in \mathsf{F}.$$

Let n be such that $G_n \subseteq K$. It follows that for every $m \in \omega$,

$$X \setminus n =^{\star} \bigcup_{n\in\omega} I_n \notin G_n,$$

which finishes the proof of 4.20. $\square$

For $K, L \in \mathsf{K}({}^{\omega}2)$ define $K \oplus L = \{X \cup Y : X \in K,\ Y \in L\}$. ($\cup$ is in $\mathbf{P}(\omega)$ the same as coordinate-wise maximum in ${}^{\omega}2$)

Let $\langle G_n : n \in \omega\rangle$ continue to be a descending sequence generating F.

Lemma 4.22. *For every $K \in \mathsf{F}$ there exists m such that $G_m \oplus G_m \subseteq K$.*

Proof. Fix $X \in \mathcal{J}$ and using the fact that $\mathcal{J}$ is a p-ideal find k such that $\{Y \in \mathcal{J} : X \setminus m \subseteq Y\}$ is cofinal in $\mathcal{J}$. The set

$$H_X = \{Y : (X \setminus k) \cup Y \in K\} \in \mathsf{F}.$$

Let $n(X)$ be such that $G_{n(X)} \subseteq H_X$. We have

$$\{X \setminus n(X)\} \oplus G_{n(X)} \subseteq K.$$

Choose n such that $\{X : n(X) = n\}$ is cofinal in $\mathcal{J}$. The set $L = \{X : \{X\setminus n\}\oplus G_n \subseteq K\} \in \mathsf{F}$. Let $m \geq n$ be such that $G_m \subseteq L$. It follows that $G_m \oplus G_m \subseteq K$. $\square$

We are ready to formulate the first result.

Theorem 4.23. *Suppose that $\mathcal{J}$ is an analytic p-ideal on ω. Then $\mathcal{J} \preceq \ell^1$. In particular, $\mathsf{add}^{\star}(\mathcal{J}) \geq \mathsf{add}(\mathcal{N})$ and $\mathsf{cof}(\mathcal{J}) \leq \mathsf{cof}(\mathcal{N})$.*

Proof. Use Lemma 4.22 to find a descending sequence $\langle G_n : n \in \omega\rangle$ generating F such that for each n,

$$\underbrace{G_{n+1} \oplus \cdots \oplus G_{n+1}}_{2^{n+1} \text{ times}} \subseteq G_n.$$

For $X \in \mathcal{J}$ let $\langle k_n(X) : n \in \omega\rangle$ be an increasing sequence such that

$$\forall n\ (X \setminus k_n(X) \in G_{n+2}).$$

Identify ω with $[\omega]^{<\omega}$ and define $\varphi_- : \mathcal{J} \longrightarrow \mathcal{C}$ and $\varphi_+ : \mathcal{C} \longrightarrow \mathcal{J}$ such that

$$X \subseteq^{\star} \varphi_+(S), \text{ whenever } \varphi_-(X) \subseteq^{\star} S.$$

Since $\mathcal{C} \equiv \ell^1 \equiv \mathcal{N}$ this will finish the proof. For $X \in \mathcal{J}$ and $n \in \omega$ define

$$\varphi_-(X)(n) = X \cap k_n(X) \in [\omega]^{<\omega} \simeq \omega.$$

Mapping φ_+ will be defined as follows. Suppose that $S \in \mathcal{C}$ is given (with $S(n) \subseteq [\omega]^{<\omega}$). For $n \in \omega$ let

$$Z_n = \{(t, s) \in S(n+1) \times S(n) : s \subseteq t \ \&\ t \setminus \max(s) \in G_{n+2}\}.$$

Now define

$$v_n = \bigcup_{(t,s)\in Z_n} t \setminus \max(s).$$

Note that v_n is a sum of at most 2^{n+1} terms, each belonging to G_{n+2}. Thus, $v_n \in G_{n+1}$ for all n.

The motivation for this definition is following: if $\varphi_-(X)(n) = X \cap k_n(X) \in S(n)$ and $\varphi_-(X)(n+1) = X \cap k_{n+1}(X) \in S(n+1)$, then

$$X \cap k_{n+1}(X) \setminus \max\big(X \cap k_n(X)\big) = X \cap \big[k_n(X), k_{n+1}(X)\big) \subseteq v_n.$$

The requirements of the definition describe this situation and filter out "background noise" coming with S.

Finally define

$$\varphi_+(S) = Y = \bigcup_n v_n.$$

By the remarks above it is clear that if $X \in \mathcal{J}$ and $S \in \mathcal{C}$ then from the fact that

$$\forall^\infty n \ \big(\varphi_-(X)(n) \in S(n)\big)$$

it follows that $X \subseteq^\star Y = \varphi_+(S)$. To finish the proof it remains to show that the range of φ^+ is contained in $\mathcal{J}$.

Let $\varphi_+(S) = Y = \bigcup_n v_n$ be defined as above. For $j \in \omega$, let $Y_j = \bigcup_{n \geq j} v_n$. Since $Y \setminus Y_j$ is finite for every j, by the lemma above, in order to show that $Y \in \mathcal{J}$ it would suffice to show that $Y_j \in G_j$.

Lemma 4.24. *For each $l \in \omega$,*

$$v_n \cup v_{n+1} \cup \cdots \cup v_{n+l} \in G_n.$$

Proof. We prove this by induction on l. For each n, $v_n \in G_{n+1}$ so the lemma is true for $l = 0$. Suppose it holds for l and all n. We have

$$v_n \cup v_{n+1} \cup v_{n+l+1} = v_n \cup (v_{n+1} \cup \dots v_{n+1+l}) \in G_{n+1} \oplus G_{n+1} \subseteq G_n,$$

which finishes the proof. $\square$

Since sets G_n are closed, we conclude that $Y_j \in G_j$. In particular, by Lemma 4.20, $Y = Y_0 \in \mathcal{J}$. $\square$

The last theorem gave us an lower bound for $\mathsf{add}^\star(\mathcal{J})$. The next theorem gives us an upper bound.

Suppose that $\mathcal{J} \subseteq \mathbf{P}(\omega)$ is an ideal. We say that $\mathcal{J}$ is *atomic* if there is $Z \in \mathcal{J}$ such that $\mathcal{J} = \{X \subseteq \omega : X \subseteq^\star Z\}$. It is clear that $\mathsf{add}^\star(\mathcal{J})$ is undefined (or equal to ∞) for an atomic ideal.

Theorem 4.25. *Suppose that $\mathcal{J}$ is an analytic p-ideal which is not atomic. Then ${}^\omega\omega \preceq \mathcal{J}$. In particular, $\mathsf{add}^\star(\mathcal{J}) \leq \mathfrak{b}$ and $\mathsf{cof}(\mathcal{J}) \geq \mathfrak{d}$.*

Proof. For $X \subseteq \omega$ let $\overline{X} \in \prod_{n\in\omega} n$ be defined as $\overline{X}(n) = |X \cap n|$. Let $\mathcal{X} = \big\{\overline{X} : X \in \mathbf{P}(\omega)\big\}$. It is easy to see that $\mathcal{X}$ is a compact space. For $\overline{X}, \overline{Y} \in \mathcal{X}$ define $\overline{X} \leq^\star \overline{Y}$ if there is a finite set $Z \subseteq \omega$ such that

$$\forall n \ \overline{X}(n) \leq \overline{Y \cup Z}(n)$$

or alternatively

$$\forall n \ \overline{X \setminus Z}(n) \leq \overline{Y}(n).$$

Lemma 4.26. $(\mathcal{J}, \mathcal{J}, \not\supseteq^\star) \preceq (\mathcal{X}, \mathcal{X}, \not\geq^\star)$.

Proof. Define $\varphi_- : \mathcal{J} \longrightarrow {}^\omega\omega$ and $\varphi_+ : {}^\omega\omega \longrightarrow \mathcal{J}$ as $\varphi_-(X) = \overline{X}$ and $\varphi_+(\overline{X}) = X$. Suppose that $\varphi_-(X) \not\geq^\star \overline{Y}$. That means that for any finite set Z,

$$\exists^\infty n \ \big(\overline{X \cup Z}(n) < \overline{Y}(n)\big).$$

It follows that $Y \not\subseteq X \cup Z$, hence $Y \not\subseteq^\star X$. $\square$

Let $\mathsf{M} \subseteq \mathsf{K}(\mathcal{X})$ be the collection of compact subsets of $\mathcal{X}$ which are downward closed (with respect to $\leq$). Let

$$\mathsf{F} = \left\{K \in \mathsf{M} : \forall X \in \mathcal{J}\ \exists n\ \left(\overline{X \setminus n} \in K\right)\right\}.$$

As before, by Lemma 4.19, F is countably generated. Let $\langle G_n : n \in \omega\rangle$ be any sequence generating F.

We will show that $(\mathcal{X}, \mathcal{X}, \not\geq^\star) \preceq ({}^\omega\omega, {}^\omega\omega, \not\geq^\star)$. We need functions $\varphi_- : \mathcal{X} \longrightarrow {}^\omega\omega$ and $\varphi_+ : {}^\omega\omega \longrightarrow \mathcal{X}$ such that for $\overline{X} \in \mathcal{X}$ and $f \in {}^\omega\omega$,

$$\overline{X} \not\leq^\star \varphi_+(f),\ \text{whenever}\ \varphi_-(\overline{X}) \not\geq^\star f.$$

Clearly, the dual morphism witnesses that ${}^\omega\omega \preceq \mathcal{X} \preceq \mathcal{J}$.

Each set G_n is a set of branches of some tree. By taking the rightmost branch (towards the larger values) at every node of the tree, we produce a countable family $\left\{\overline{Z}^n_m : m \in \omega\right\}$ such that

$$\forall X \in \mathcal{J}\ \exists m\ \left(\overline{X} \leq^\star \overline{Z}^n_m\right).$$

Since $\mathcal{J}$ is a p-ideal, for each n there is m such that

$$\left\{X \in \mathcal{J} : \overline{X} \leq^\star \overline{Z}^n_m\right\} \text{ is cofinal in } \mathcal{J}.$$

Denote the first such $\overline{Z}^n_m$ by $\overline{U}_n$. Without loss of generality we can assume $\overline{U}_n \geq \overline{U}_{n+1}$ for all n.

We have the following two cases:

Case 1. There exists $\overline{Z} \in \mathcal{X}$ such that

1. $\overline{X} \leq^\star \overline{Z}$ for every $X \in \mathcal{J}$.
2. $\overline{Z} \leq^\star \overline{U}_n$ for every n.

In this case $\mathcal{J}$ is atomic. Note that condition (2) implies that

$$\forall n\ \exists m\ \left(\overline{Z \setminus m} \leq \overline{U}_n \in G_n\right).$$

Thus by Lemma 4.20, $Z \in \mathcal{J}$. Condition (1), together with the fact that $\mathcal{J}$ is an ideal, implies that $X \subseteq^\star Z$ for every $X \in \mathcal{J}$.

Case 2. Suppose that there is no $\overline{Z}$ as in Case 1

For $X \in \mathcal{J}$ and $n \in \omega$ define

$$\varphi_-(\overline{X})(n) = \max\left\{\max\left\{j : \overline{X}(j) > \overline{U}_i(j)\right\} : i \leq n\right\}.$$

For $f \in {}^\omega\omega$ let

$$\varphi_+(f) \text{ be any } X \in \mathcal{J} \text{ such that } f \leq^\star \varphi_-(\overline{X}).$$

It is clear that these mappings have the required properties provided that they are correctly defined. Thus, the following lemma will complete the proof:

Lemma 4.27. *For every $f \in {}^\omega\omega$ there exists an $X \in \mathcal{J}$ such that $f \leq^\star \varphi_-(\overline{X})$.*

Proof. Suppose not and let $f \in {}^\omega\omega$ be a strictly increasing function such that for every set $X \in \mathcal{J}$ the set

$$Z_X = \left\{n : \varphi_-(\overline{X})(n) > f(n)\right\}$$

is coinfinite. It follows that the family $\{Z_X : X \in \mathcal{J}\}$ generates a proper analytic ideal $\mathcal{H}$. As before, $\mathcal{H}$ has the Baire property, hence by Theorem 4.21(3), there exists a partition $\{I_n : n \in \omega\}$ such that

$$\forall X \in \mathcal{J}\ \forall^\infty n\ \big(I_n \not\subseteq Z_X\big).$$

Let $h(n) = f\big(\max(I_{n+1})\big)$ for $n \in \omega$ and consider the function

$$\overline{U} = \bigcup_n \overline{U}_n \restriction \big[h(n), h(n+1)\big).$$

Clearly, $\overline{U}_n \geq^\star \overline{U}$ for all n. We will show that $\overline{X} \leq^\star \overline{U}$ for $X \in \mathcal{J}$, which will give the contradiction.

Fix $X \in \mathcal{J}$. Suppose that $n \in I_m$ and $k \in I_{m+1} \setminus Z_X$. Clearly, $m \leq n$ and $k \leq \max\big(I_{m+1}\big)$. We have

$$\max\big\{j : \overline{X}(j) > \overline{U}_n(j)\big\} \leq \varphi_-(X)(k) \leq f(k) \leq f\big(\max(I_{m+1})\big) \leq h(n).$$

In particular,

$$\forall j > h(n) \overline{X}(j) \leq \overline{U}_n(j).$$

It follows that $\overline{X} \leq^\star \overline{U}$. □

□

Historical remarks Theorem 4.1 was proved in [34] and [7]. Theorem 4.6 is due to Pawlikowski and Recław in their [34]. Theorems 4.5 and 4.7 were proved in [3]. Theorem 4.8 was proved in [34]. The second part is due to Miller [29]. The first part of Theorem 4.11 was proved in [34] and the second in [2]. Todorcevic proved Theorem 4.25 [50]. I learned Theorem 4.23 from Todorcevic [49]. The result can also be attributed to Louveau and Velickovic (see their [28], Theorem 5). Methods used in the proof, in particular Lemmas 4.17 and 4.22 are due to Solecki [45] and [46]. Similar ideas were already present in [50] and earlier in [21]. Theorem 4.18 is due to Christensen and Saint Raymond. It was generalized in [25]. Theorem 4.21 was proved by Talagrand.

5. Cofinality of $\mathsf{cov}(\mathcal{J})$ and $\mathsf{COV}(\mathcal{J})$

It is clear that cardinal invariants add, non and cof have uncountable cofinality and families ADD, NON and COF are σ-ideals. It this section we investigate cov and COV for both ideals $\mathcal{M}$ and $\mathcal{N}$.

Theorem 5.1. $\mathsf{COV}(\mathcal{M})$ *is a σ-ideal. In particular,* $\mathsf{cf}\big(\mathsf{cov}(\mathcal{M})\big) > \aleph_0$.

Proof. Suppose that $\{X_n : n \in \omega\} \subseteq \mathsf{COV}(\mathcal{M})$. Let $x \rightsquigarrow f_x \in {}^\omega\omega$ be a Borel mapping. It is enough to find $g \in {}^\omega\omega$ such that

$$\forall n\ \forall x \in X_n\ \exists^\infty m\ \big(g(m) = f_x(m)\big).$$

Let $\{A_k : k \in \omega\}$ be a partition of ω into infinitely many infinite pieces. For each n consider the mapping $x \rightsquigarrow f_x \restriction A_n$ and find $g_n \in {}^{A_n}\omega$ such that

$$\forall x \in X_n\ \exists^\infty k \in A_n\ \big(f_x(k) = g_n(k)\big).$$

Then $g = \bigcup_n g_n$ is as required. □

In the presence of many dominating reals we have a similar result for the measure ideal.

Theorem 5.2. *If* $\mathsf{cov}(\mathcal{N}) \leq \mathfrak{b}$ *then* $\mathsf{cf}\big(\mathsf{cov}(\mathcal{N})\big) > \aleph_0$.

Proof. See [4] of [8]. $\square$

The following surprising result of Shelah shows that without any additional assumptions it is not possible to show that $\mathsf{cov}(\mathcal{N})$ has uncountable cofinality.

Theorem 5.3. *It is consistent with* ZFC *that* $\mathsf{COV}(\mathcal{N})$ *is not a* σ*-ideal and* $\mathsf{cf}\big(\mathsf{cov}(\mathcal{N})\big) = \aleph_0$.

The proof of this theorem will occupy the rest of this section. The model will be obtained by a two-step finite support iteration. We start with a suitably chosen model $\mathbf{V}_0$ satisfying $2^{\aleph_0} = \aleph_1$ and add $\aleph_\omega$ Cohen reals followed by a finite support iteration of subalgebras of the random algebra $\mathbf{B}$. We start by developing various tools needed for the construction.

He random real algebra. Recall that the random real algebra can be represented as

$$\mathbf{B} = \{P \subseteq {}^\omega 2 : \mu(P) > 0 \text{ and } P \text{ is closed}\}.$$

For $P_1, P_2 \in \mathbf{B}$, $P_1 \geq P_2$ if $P_1 \subseteq P_2$. Elements of $\mathbf{B}$ can be coded by reals in the following way. Let $\widetilde{P} \in \mathbf{V}_0$ be a universal closed set, i.e. $\widetilde{P} \subseteq {}^\omega 2 \times {}^\omega 2$ is Borel and for every closed set $P \subseteq {}^\omega 2$ there is x such that $P = (\widetilde{P})_x$. Let $H = \Big\{x : \mu\big((\widetilde{P})_x\big) > 0\Big\}$. By Theorem 3.6(1), H is a Borel set. Define $\widetilde{B} = (H \times {}^\omega 2) \cap \widetilde{P}$. If M is a model of ZFC then we define

$$\mathbf{B}^M = \Big\{P \in \mathbf{B} : \exists x \in M \cap {}^\omega 2 \ \big(P = (\widetilde{B})_x\big)\Big\}.$$

Δ-systems. The following concepts will be crucial for the construction of the model.

Definition 5.4. *Let* $\mathcal{R} \in \mathbf{V}_0$ *be a forcing notion Suppose that* $\bar{p} = \langle p_n : n \in \omega\rangle$ *is a sequence of conditions in* $\mathcal{R}$. *Let* $\dot{X}_{\bar{p}}$ *be the* $\mathcal{R}$*-name for the set* $\{n : p_n \in \dot{G}_{\mathcal{R}}\}$. *In other words, for every* n, $p_n = [\![n \in \dot{X}_{\bar{p}}]\!]$.

At the moment we will be concerned with the case when $\mathcal{R} = \mathbf{C}_{\aleph_{\omega+1}}$ is the forcing notion adding $\aleph_{\omega+1}$ Cohen reals.

Definition 5.5. *Let* $\Delta \subseteq \big[\mathbf{C}_{\aleph_{\omega+1}}\big]^\omega$ *be the collection of all sequences* $\bar{p} = \{p_n : n \in \omega\}$ *such that there exists* $k, l \in \omega$ *and* $g \in {}^{l\times\omega}\omega$, $s \in {}^k\omega$ *such that*

1. $\mathsf{dom}(p_n) = \{\beta_1, \dots, \beta_k\} \dot\cup \{\alpha_1^n, \dots, \alpha_l^n\}$, *with* $\beta_1 < \dots < \beta_k$ *and* $\alpha_1^n < \dots < \alpha_l^n$ *for* $n \in \omega$ *(so the* $\mathsf{dom}(p_n)$*'s form a* Δ*-system with root* $\{\beta_1, \dots, \beta_k\}$*),*
2. $\alpha_i^n < \alpha_i^m$ *for* $n < m$,
3. $p_n(\alpha_i^n) = g(i, n)$ *for every* $i \leq l, n \in \omega$,
4. $p_n(\beta_i) = s(i)$ *for* $i \leq k, n \in \omega$.

Let $p_{\bar{p}} = p_0 {\restriction} \{\beta_1, \dots, \beta_k\}$.

Note that if $\bar{p} \in \Delta$ then $f_{\bar{p}} = \bigcup_{n\in\omega} p_n$ is a function. Moreover, $p_{\bar{p}} = f_{\bar{p}} {\restriction} \{\beta_1, \dots, \beta_k\}$ and $p_{\bar{p}} \Vdash_{\mathbf{C}_{\aleph_{\omega+1}}} X_{\bar{p}}$ is infinite.

Definition 5.6. *A subset* $\Delta' \subseteq \Delta$ *is filter-like if for any* $\bar{p}^1, \dots, \bar{p}^n \in \Delta'$ *there exists* q *such that*

$$q \Vdash_{\mathbf{C}_{\aleph_{\omega+1}}} \bigcap_{i\leq n} X_{\bar{p}^i} \text{ is infinite.}$$

Theorem 5.7. *Suppose that* $\mathbf{V} \models 2^{\aleph_0} = \aleph_1$ & $2^{\aleph_1} = \aleph_{\omega+1}$. *Then* Δ *is the union of* $\aleph_1$ *filter-like sets.*

Proof. Let T be the collection of $\langle k, l, v, \{f_{i,n}, g_j : i \leq l, j \leq k, n \in \omega\}, g, s\rangle$ such that

1. $k, l \in \omega$,
2. $v \in [\aleph_1]^{\leq \aleph_0}$,
3. $g_j, f_{i,n} \in {}^v\omega$ are pairwise different for $i \leq l, j \leq k, n \in \omega$,
4. $g \in {}^{l\times\omega}\omega$,
5. $s \in {}^k\omega$.

From the assumption about the cardinal arithmetic in $\mathbf{V}$ it follows that $\mathbf{V} \models \aleph_n^{\aleph_0} = \aleph_n$ for $n \geq 1$. In particular $\mathbf{V} \models |T| = \aleph_1$. Moreover, since $\mathbf{V} \models 2^{\aleph_1} = \aleph_{\omega+1}$ we can find in $\mathbf{V}$ an enumeration $\langle h_\alpha : \alpha < \aleph_{\omega+1}\rangle$ of $2^{\aleph_1}$.

Given $t = \langle k, l, v, \{f_{i,n}, g_j : i \leq l, j \leq k, n \in \omega\}, g, s\rangle \in T$ define $\Delta_t \subseteq \Delta$ to be collection of all $\bar{p} = \langle p_n : n \in \omega\rangle$ such that

1. $\mathsf{dom}(p_n) = \{\beta_1 < \cdots < \beta_k\} \dot{\cup} \{\alpha_1^n < \cdots < \alpha_l^n\}$,
2. $p_n(\alpha_i^n) = g(i, n)$,
3. $p_n(\beta_i) = s(i)$,
4. $\forall i \leq l \ \big(h_{\alpha_i^n}\restriction v = f_{i,n}\big)$,
5. $\forall j \leq k \ \big(h_{\beta_j}\restriction v = g_j\big)$.

Lemma 5.8. Δ_t *is filter-like for every* $t \in T$.

Proof. Suppose that $\bar{p}^1, \bar{p}^2 \in \Delta_t$. First we show that $f_{\bar{p}^1} \cup f_{\bar{p}^2}$ is a function. Suppose that $\alpha \in \mathsf{dom}(f_{\bar{p}^1}) \cap \mathsf{dom}(f_{\bar{p}^2})$. Consider the function h_α and note that exactly one of the following possibilities happens:

1. there exists exactly one pair (n, i) such that $h_\alpha\restriction v = f_{i,n}$. In this case $f_{\bar{p}^1}, f_{\bar{p}^2}$ agree on α with the value $g(i, n)$,
2. there exists exactly one $j \leq k$ such that $h_\alpha\restriction v = g_j$ (so $f_{\bar{p}^1}(\alpha) = f_{\bar{p}^2}(\alpha) = s(j)$).

Now, put $q = p_{\bar{p}_1} \cup p_{\bar{p}_2}$ and note that q has the required property. □

To finish the proof of 5.7 note that $\Delta = \bigcup_{t\in T} \Delta_t$. Suppose that $\bar{p} = \langle p_n : n \in \omega\rangle \in \Delta$. Let k, l, g and s be as in 5.5, and put v to be a countable set such that $h_{\alpha_i^n}\restriction v$ and $h_{\beta_j}\restriction v$ are pairwise different. □

Finitely additive measures on ω.

Definition 5.9. *A set* $\mathcal{A} \subseteq \mathbf{P}(\omega)$ *is an algebra if*

1. $X \cup Y \in \mathcal{A}$ *whenever* $X, Y \in \mathcal{A}$,
2. $\omega \setminus X \in \mathcal{A}$ *whenever* $X \in \mathcal{A}$,
3. $\emptyset, \omega \in \mathcal{A}$, $\{n\} \in \mathcal{A}$ *for* $n \in \omega$.

Given an algebra $\mathcal{A}$, *a function* $m : \mathcal{A} \longrightarrow [0, 1]$ *is a finitely additive measure if*

1. $m(\omega) = 1$ *and* $m(\emptyset) = m(\{n\}) = 0$ *for every* n,
2. *if* $X, Y \subseteq \omega$ *are disjoint, then* $m(X \cup Y) = m(X) + m(Y)$.

Any non-principal filter on ω corresponds to a finitely additive measure and any ultrafilter is a maximal such measure.

Definition 5.10. *For a real valued function $f : \omega \longrightarrow [0,1]$ define*

$$\int_\omega f \; dm = \lim_{n\to\infty} \sum_{k=0}^{2^n} \frac{k}{2^n} \cdot m(A_k),$$

where

$$A_k = \left\{ n : \frac{k}{2^n} \leq f(n) < \frac{k+1}{2^n} \right\}.$$

We leave it to the reader to verify that integration with respect to m has its usual properties.

The following is the special case of the Hahn–Banach theorem.

Theorem 5.11 (Hahn–Banach)**.** *Suppose that m is a finitely additive measure on an algebra $\mathcal{A}$, and $X \notin \mathcal{A}$. Let $a \in [0,1]$ be such that*

$$\sup\{m(A) : A \subseteq X, A \in \mathcal{A}\} \leq a \leq \inf\{m(B) : X \subseteq B, B \in \mathcal{A}\}.$$

Then there exists a measure $\bar{m}$ on $\mathbf{P}(\omega)$ extending m such that $\bar{m}(X) = a$.

We will need several results concerning the existence of measures in forcing extensions.

Lemma 5.12. *Let $m_0 \in \mathbf{V}$ be a finitely additive measure on ω. For $i = 1, 2$ let $\mathcal{R}_i$ be a forcing notion and let $\dot{m}_i$ be a $\mathcal{R}_i$-name for a finitely additive measure on $\mathbf{V}^{\mathcal{R}_i} \cap \mathbf{P}(\omega)$ extending m_0. Then there exists a $\mathcal{R}_1 \times \mathcal{R}_2$-name for a measure $\dot{m}_2$ extending both $\dot{m}_1$ and $\dot{m}_2$.*

Proof. We extend the measures using the Hahn-Banach theorem and we only need to check that the requirements are consistent. Suppose that we have $\mathcal{R}_1$-name $\dot{X}$ and $\mathcal{R}_2$-name $\dot{Y}$ such that $\Vdash_{\mathcal{R}_1\times\mathcal{R}_2} \dot{X} \subseteq^\star \dot{Y}$. A necessary and sufficient condition for both measures to have a common extension is that in such a case $m_1(\dot{X}) \leq m_2(\dot{Y})$. Let $(\bar{p}, \bar{q}) \in \mathcal{R}_1 \times \mathcal{R}_2$ and $\bar{n}$ be such that

$$(\bar{p}, \bar{q}) \Vdash_{\mathcal{R}_1\times\mathcal{R}_2} \dot{X} \setminus \bar{n} \subseteq \dot{Y}.$$

Let

$$Z = \Big\{ n > \bar{n} : \exists p \in \mathcal{R}_1 \, \Big(p \geq \bar{p} \;\&\; p \Vdash_{\mathcal{R}_1} n \in \dot{X} \Big) \Big\}.$$

Set Z belongs to $\mathbf{V}$ and $\bar{p} \Vdash_{\mathcal{R}_1} \dot{X} \setminus \bar{n} \subseteq Z$. Similarly $\bar{q} \Vdash_{\mathcal{R}_2} Z \subseteq \dot{Y}$. In particular,

$$(\bar{p}, \bar{q}) \Vdash_{\mathcal{R}_1\times\mathcal{R}_2} \dot{m}_1(\dot{X}) \leq \dot{m}_1(Z) = m_0(Z) = \dot{m}_2(Z) \leq \dot{m}_2(\dot{Y}).$$

□

We will need the following theorem.

Theorem 5.13. *Suppose that $m \in \mathbf{V}$ is a finitely additive atomless measure on ω and $v \in \mathbf{B}$. For a $\mathbf{B}$-name $\dot{X}$ for an element of $[\omega]^\omega$ define*

$$\dot{m}^v_{\mathbf{B}}(\dot{X}) = \sup\left\{ \inf\left\{ \int_\omega \frac{\mu(q \cap [\![n \in \dot{X}]\!])}{\mu(q)} \; dm : q \geq p \right\} : p \geq v, \; p \in \dot{G}_{\mathbf{B}} \right\}.$$

The name $\dot{m}^v_{\mathbf{B}}$ has the following properties:

1. $v \Vdash_{\mathbf{B}} \dot{m}^v_{\mathbf{B}} : \mathbf{P}(\omega) \longrightarrow [0,1] = \{x \in \mathbb{R} : 0 \leq x \leq 1\}$,
2. $v \Vdash_{\mathbf{B}} \dot{m}^v_{\mathbf{B}}$ *is a finitely additive atomless measure,*
3. *for* $X \in \mathbf{V} \cap \mathbf{P}(\omega)$ $v \Vdash_{\mathbf{B}} \dot{m}^v_{\mathbf{B}}(X) = m(X)$,

4. *if $\dot X$ is a $\mathbf{B}$-name for a subset of ω and $\mu\left(\llbracket n\in\dot X\rrbracket_{\mathbf{B}}\cap v\right)/\mu(v)=a>0$ for all n, then there is a condition $p\in\mathbf{B}$, $p\geq v$ such that $p\Vdash_{\mathbf{B}}\dot m^v_{\mathbf{B}}(\dot X)\geq a$.*

Proof. Without loss of generality we can assume that $v=2^\omega$ and therefore we will drop the superscript v altogether.

(1) is clear.

(2) For a $\mathbf{B}$-name $\dot X$ for a subset of ω and $p\in\mathbf{B}$ let

$$m_p(\dot X)=\int_\omega \frac{\mu\big(p\cap\llbracket n\in\dot X\rrbracket\big)}{\mu(p)}\;dm$$

and

$$m^\star_p(\dot X)=\inf\{m_q(\dot X):q\geq p\}.$$

Clearly, $\dot m_{\mathbf{B}}(\dot X)=\sup_{p\in\dot G}\inf_{q\geq p}m_q(\dot X)=\sup_{p\in\dot G}m^\star_p(\dot X)$. Note that if $p\Vdash_{\mathbf{B}}\dot X\subseteq\dot Y$ then $p\cap\llbracket n\in\dot X\rrbracket\subseteq p\cap\llbracket n\in\dot Y\rrbracket$ for every n. It follows that $m_p(\dot X)\leq m_p(\dot Y)$ and $m^\star_p(\dot X)\leq m^\star_p(\dot Y)$.

Similarly, if $p\Vdash_{\mathbf{B}}\dot X\cap\dot Y=\emptyset$ and $\dot Z$ is a name for $\dot X\cup\dot Y$ then $m_p(\dot X)+m_p(\dot Y)=m_p(\dot Z)$ and $m^\star_p(\dot X)+m^\star_p(\dot Y)\leq m^\star_p(\dot Z)$.

Lemma 5.14. *$p\Vdash_{\mathbf{B}}\dot m_{\mathbf{B}}(\dot X)\geq r\iff m^\star_p(\dot X)\geq r$.*

Proof. $\leftarrow$ is obvious.

$\rightarrow$. Suppose that $p\Vdash_{\mathbf{B}}\dot m_{\mathbf{B}}(\dot X)\geq r$. Fix any real $t<r$ and $p'\geq p$. It follows that

$$D=\{q\geq p':m^\star_q(\dot X)\geq t\}$$

is dense below p'. Let $\{q_n:n\in\omega\}$ be a maximal antichain in D. We have $m_{q_n}(\dot X)\geq m^\star_{q_n}(\dot X)\geq t$. It follows that $\mu(q_n\cap\llbracket n\in\dot X\rrbracket)\geq t\cdot\mu(q_n)$. Since $p'=\bigcup_n q_n$ and q_n are pairwise disjoint, we get

$$\mu(p'\cap\llbracket n\in\dot X\rrbracket)\geq t\cdot\mu(p').$$

We conclude that $m^\star_p(\dot X)\geq t$ and since t was arbitrary, $m^\star_p(\dot X)\geq r$. □

Now we show that $\dot m_{\mathbf{B}}$ is a finitely additive measure.

Suppose that $\Vdash_{\mathbf{B}}\dot X\subseteq\dot Y$. Suppose that $p\Vdash_{\mathbf{B}}\dot m_{\mathbf{B}}(\dot X)>\dot m_{\mathbf{B}}(\dot Y)$. Let $q\geq p$ and r be such that $q\Vdash_{\mathbf{B}}\dot m_{\mathbf{B}}(\dot X)>r\geq\dot m_{\mathbf{B}}(\dot Y)$. Then $m^\star_q(\dot X)\geq r$ and $m^\star_q(\dot Y)<r$ – contradiction.

Suppose that $\Vdash_{\mathbf{B}}\dot X\cap\dot Y=\emptyset$ and let $\dot Z$ be a name for $\dot X\cup\dot Y$. Let p, r_1,r_2 be such that $p\Vdash_{\mathbf{B}}\dot m_{\mathbf{B}}(\dot X)\geq r_1$ and $p\Vdash_{\mathbf{B}}\dot m_{\mathbf{B}}(\dot Y)\geq r_1$. It follows that $m^\star_p(\dot X)\geq r_1$ and $m^\star_p(\dot Y)\geq r_2$. Thus $m^\star_p(\dot Z)\geq r_1+r_2$, so $p\Vdash_{\mathbf{B}}\dot m_{\mathbf{B}}(\dot Z)\geq\dot m_{\mathbf{B}}(\dot X)+\dot m_{\mathbf{B}}(\dot Y)$.

Suppose that $p\Vdash_{\mathbf{B}}\dot m_{\mathbf{B}}(\dot Z)>\dot m_{\mathbf{B}}(\dot X)+\dot m_{\mathbf{B}}(\dot Y)$. There are reals r_1,r_2 and $q\geq p$ such that $q\Vdash_{\mathbf{B}}\dot m_{\mathbf{B}}(\dot X)<r_1$, $q\Vdash_{\mathbf{B}}\dot m_{\mathbf{B}}(\dot Y)<r_2$ and $q\Vdash_{\mathbf{B}}\dot m_{\mathbf{B}}(\dot Z)>r_1+r_2$. Use 5.14, to find $q'\geq q$ such that $m_{q'}(\dot X)<r_1$ and $m_{q'}(\dot Y)<r_2$. By 5.14, $m_{q'}(\dot Z)\geq m^\star_{q'}(\dot Z)\geq r_1+r_2$. On the other hand, since $m_{q'}$ is additive, $m_{q'}(\dot Z)<r_1+r_2$ – contradiction.

(3) Suppose that $\dot{X}$ is a $\mathbf{B}$-name and for some $p \in \mathbf{B}$ and $X \in \mathbf{V} \cap \mathbf{P}(\omega)$, $p \Vdash_{\mathbf{B}} \dot{X} = X$. That means that for every $q \geq p$,

$$\frac{\mu(q \cap [\![n \in \dot{X}]\!])}{\mu(q)} = \begin{cases} 1 & \text{if } n \in X \\ 0 & \text{if } n \notin X \end{cases} .$$

It follows that $p \Vdash_{\mathbf{B}} \dot{m}_{\mathbf{B}}(X) \geq m(X)$. Since $\dot{m}_{\mathbf{B}}$ is a measure, by looking at the complements we get, $p \Vdash_{\mathbf{B}} 1 - \dot{m}_{\mathbf{B}}(X) \geq 1 - m(X)$, hence $p \Vdash_{\mathbf{B}} \dot{m}_{\mathbf{B}}(X) = m(X)$.

(4) Suppose that $\mu\left([\![n \in \dot{X}]\!]_{\mathbf{B}}\right) = a > 0$ for $n \in \omega$. Let

$$D = \{p : \exists \varepsilon > 0 \ m_p(\dot{X}) \leq (1-\varepsilon) \cdot a\}.$$

If D is not dense in $\mathbf{B}$, then the condition witnessing that has the required property.

So suppose that D is dense and work towards a contradiction. Let $\{q_n : n \in \omega\}$ be a maximal antichain in D. Clearly $\sum_{n=0}^{\infty} \mu(q_n) = 1$. Let $\varepsilon_0 > 0$ be such that $m_{q_0}(\dot{X}) \leq (1-\varepsilon_0) \cdot a$, which means that

$$\int_\omega \mu\left(q_0 \cap [\![n \in \dot{X}]\!]\right) \, dm \leq (1-\varepsilon_0) \cdot a \cdot \mu(q_0).$$

Similarly for $n > 0$,

$$\int_\omega \mu\left(q_n \cap [\![n \in \dot{X}]\!]\right) \, dm \leq a \cdot \mu(q_n).$$

Let $q = \bigcup_{i \leq n} q_n$. We have

$$\int_\omega \mu\left(q \cap [\![n \in \dot{X}]\!]\right) \, dm \leq (1-\varepsilon_0) \cdot a \cdot \mu(q_0) + \sum_{j=1}^{n} a \cdot \mu(q_j) = a \cdot \mu(q) - \varepsilon_0 \cdot a \cdot \mu(q_0).$$

This is a contradiction since

$$\lim_{\mu(q) \to 1} \int_\omega \mu\left(q \cap [\![n \in \dot{X}]\!]\right) \, dm = a.$$

$\square$

The iteration

Let $\mathbf{V}_0$ be a model satisfying $2^{\aleph_0} = \aleph_1$ and $2^{\aleph_1} = 2^{\aleph_2} = \cdots = \aleph_{\omega+1}$. In $\mathbf{V}_0$ we will define the following objects:

1. A finite support iteration $\langle \mathcal{P}_\alpha, \dot{\mathcal{Q}}_\alpha : \alpha < \aleph_{\omega+1} \rangle$.
2. A sequence $\langle A_\alpha : \aleph_\omega \leq \alpha < \aleph_{\omega+1} \rangle$.
3. A sequence $\langle \dot{m}^\xi_\alpha : \aleph_\omega \leq \alpha < \aleph_{\omega+1}, \xi < \aleph_1 \rangle$ such that
   (a) $\dot{m}^\xi_\alpha$ is a $\mathcal{P}_\alpha$-name for a finitely additive measure on ω,
   (b) $\dot{m}^\xi_\alpha$ extends $\bigcup_{\beta<\alpha} \dot{m}^\xi_\beta$,
   (c) if $\mathsf{cf}(\gamma) > \aleph_0$ then $\dot{m}^\xi_\gamma = \bigcup_{\beta<\gamma} \dot{m}^\xi_\beta$.

The definition is inductive. Formally, given $\mathcal{P}_\alpha$, $\left\{\dot{m}^\xi_\alpha : \xi < \aleph_1\right\}$ and A_α we define $\left\{\dot{m}^\xi_{\alpha+1} : \xi < \aleph_1\right\}$ followed by $A_{\alpha+1}$ and then $\mathcal{P}_{\alpha+1} = \mathcal{P}_\alpha \star \dot{\mathcal{Q}}_\alpha$.

For limit α, $\mathcal{P}_\alpha$ and $\{\dot{m}^\xi_\alpha : \xi < \aleph_1\}$ will be defined by the previous values and $A_\alpha = \emptyset$. Since the definition of $\dot{m}^\xi_\alpha$ is most complicated it is more natural to proceed in the reverse order by making commitments about the defined objects as we go along.

We will use the following notation: suppose that $\langle \mathcal{P}_\alpha, \dot{\mathcal{Q}}_\alpha : \alpha < \delta\rangle$ is a finite support iteration and $A \subseteq \delta, A \in \mathbf{V}_0$. Let $\mathcal{P}(A)$ be the subalgebra generated by $\dot{G}{\restriction}A$ and let $\mathbf{V}_0[\dot{G}{\restriction}A]$ denote model $\mathbf{V}_0[\dot{G} \cap \mathcal{P}(A)]$. Note that if $|A| = \aleph_n$, $n > 0$ then $\mathbf{V}_0[\dot{G}{\restriction}A] \models 2^{\aleph_0} = \aleph_n$.

To define the iteration we require that:

A0. $A_\alpha \subseteq \alpha$ for $\alpha < \aleph_{\omega+1}$.

Let $\langle \mathcal{P}_\alpha, \mathcal{Q}_\alpha : \alpha < \aleph_{\omega+1}\rangle$ be a finite support iteration such that

$$\Vdash_\alpha \mathcal{Q}_\alpha = \begin{cases} \mathbf{C} & \text{if } \alpha < \aleph_\omega \\ \mathbf{B}^{\mathbf{V}_0[\dot{G}\restriction A_\alpha]} & \text{if } \alpha \geq \aleph_\omega \end{cases} .$$

Lemma 5.15. *Suppose that G is $\mathcal{P}_\alpha$-generic over $\mathbf{V}_0$ and $x \in \mathbf{V}_0[G] \cap \mathbf{P}(\omega)$. Then x that can be computed from countably many generic reals with indices in A. In other words, there exists a countable set $A \subseteq \alpha$, $A \in \mathbf{V}_0$ and a Borel function $f \in {}^\omega({}^\omega 2) \longrightarrow {}^\omega 2$, $f \in \mathbf{V}_0$ and a set $\{\alpha_n : n \in \omega\} \in \mathbf{V}_0 \cap [A]^\omega$ such that $x = f\big(\dot{G}(\alpha_1), \ldots, \dot{G}(\alpha_n), \ldots\big)$.*

Proof. Induction on α.

Case 1 $\alpha = \beta + 1$. Let $G \subseteq \mathcal{P}_\alpha$ be a generic filter and let $x \in \mathbf{V}_0[G]$. Work in the model $\mathbf{V}_0[G{\restriction}\alpha]$. Since $\mathcal{P}_\alpha = \mathcal{P}_\beta \star \mathbf{B}^{\mathbf{V}_0[\dot{G}\restriction A_\beta]}$ there exists a Borel function $\tilde{f} \in \mathbf{V}_0[G \cap \mathcal{P}_\beta]$ such that

$$\mathbf{V}_0[G \cap \mathcal{P}_\beta] \models \tilde{f}\big(G(\alpha)\big) = x.$$

Since $\tilde{f}$ is coded by a real, there exists a set $A = \{\alpha_n : n \in \omega\} \subseteq \beta$ and a function $f \in \mathbf{V}_0$ such that

$$\tilde{f} = f\big(G(\alpha_1), \ldots, G(\alpha_n), \ldots\big).$$

Function f and the set $A \cup \{\beta\}$ are the objects we are looking for.

Case 2 $\mathsf{cf}(\alpha) = \aleph_0$. Fix an increasing sequence $\langle \alpha_n : n \in \omega\rangle$ such that $\sup_n \alpha_n = \alpha$ and suppose that x is a $\mathcal{P}_\alpha$-name for a real number (i.e. a set of countably many antichains. Let x_n be a $\mathcal{P}_{\alpha_n}$-name for a real obtained by restriction conditions in these antichains to α_n. Note that $\Vdash_{\mathcal{P}_\alpha} \lim_n x_n = x$. Apply the induction hypothesis to x_n's to get Borel functions f_n and countable sets A_n. Let $A = \bigcup_n A_n$ and let $f : {}^{\omega\times\omega}({}^\omega 2) \longrightarrow {}^\omega 2$ be defined as

$$f(\ldots, x^n_m, \ldots) = \lim_n f_n(\ldots, x^n_m, \ldots).$$

Case 3 $\mathsf{cf}(\alpha) > \aleph_0$. Since no reals are added at the step α there is nothing to prove. □

Furthermore, we will require that

A1. $|A_\alpha| < \aleph_\omega$, for any $\aleph_\omega \leq \alpha < \aleph_{\omega+1}$.

A2. For every set $A \in [\aleph_{\omega+1}]^{<\aleph_\omega}$ there are cofinally many α with $A \subseteq A_\alpha$.

To state the next requirement we will need the following notation: suppose that $A \subseteq \aleph_{\omega+1}$. Let $\mathcal{P}{\restriction}A = \{p \in \mathcal{P} : \mathsf{dom}(p) \subseteq A\}$. Suppose that $\dot{f} \subseteq {}^\omega 2 \times {}^\omega 2$ is a name for an arbitrary function from ${}^\omega 2$ to ${}^\omega 2$ (not necessarily Borel). Then $\dot{f}{\restriction}A = \{(\dot{x}, \dot{y}) \in \dot{f} : \dot{x}, \dot{y} \text{ are } \mathcal{P}{\restriction}A\text{-names}\}$.

A3. $\mathsf{dom}(\dot{m}^\xi_\alpha{\restriction}A_\beta) = \mathbf{P}(\omega) \cap \mathbf{V}_0[\dot{G}{\restriction}A_\beta]$ for every $\xi < \aleph_1$ and $\aleph_\omega \leq \beta \leq \alpha < \aleph_{\omega+1}$. In other words, $\dot{m}^\xi_\alpha{\restriction}A_\beta$ is a name for finitely additive measure on $\mathbf{P}(\omega) \cap \mathbf{V}_0[\dot{G}{\restriction}A_\beta]$.

Suppose that $\{\dot{m}^\xi_\delta : \xi < \aleph_1\}$ is given.

Assume that $\delta = \alpha + 1$ and that in order to meet the requirement A2 we have to cover a certain set A of size $\aleph_n$. Define a sequence $\langle A^\gamma_{\alpha+1} : \gamma < \omega_1\rangle$ such that

1. $A^0_{\alpha+1} = A$,
2. $A^\beta_{\alpha+1} \subseteq A^\delta_{\alpha+1}$ for $\beta \leq \delta$,
3. $A^\delta_{\alpha+1} = \bigcup_{\beta<\delta} A^\beta_{\alpha+1}$ for limit δ,
4. for every set $X \in \mathbf{V}_0[\dot{G}{\restriction}A^\beta_{\alpha+1}]$ and $\xi < \aleph_1$, $\dot{m}^\xi_{\alpha+1}(X) \in \mathbf{V}_0[\dot{G}{\restriction}A^{\beta+1}_{\alpha+1}]$,
5. $|A^\gamma_{\alpha+1}| = \aleph_n + \aleph_1$ for all γ.

Note that since $\mathbf{V}_0[\dot{G}{\restriction}A^\beta_{\alpha+1}] \models 2^{\aleph_0} = \aleph_n$, in order to produce $A^{\beta+1}_{\alpha+1}$ we have to add to $A^\beta_{\alpha+1}$ at most $\aleph_n + \aleph_1$ countable sets. Finally let $A_{\alpha+1} = \bigcup_{\gamma<\omega_1} A^\gamma_{\alpha+1}$. It is clear that $A_{\alpha+1}$ is as required.

If δ is limit then we put $A_\delta = \emptyset$. Note that in both cases condition A3 is satisfied by the induction hypothesis and the fact that $\dot{m}^\xi_\delta$ extends $\bigcup_{\alpha<\delta} \dot{m}^\xi_\alpha$.

In order to finish the construction we have to define measures $\{\dot{m}^\xi_\alpha : \aleph_\omega \leq \alpha < \aleph_{\omega+1}\}$.

We start with the definition of a certain dense subset of $\mathcal{P}$ and from now on use only conditions belonging to this subset. Let $D \subseteq \mathcal{P}$ be a subset such that $p \in D$ if

1. $\mathsf{dom}(p) \in [\aleph_{\omega+1}]^{<\omega}$,
2. $p(\alpha) \in {}^{<\omega}\omega \simeq \mathbf{C}$, for $\alpha \in \mathsf{dom}(p) \cap \aleph_\omega$,
3. for each $\alpha \in \mathsf{dom}(p) \setminus \aleph_\omega$,
   (a) $\Vdash_\alpha p(\alpha) \in \mathbf{B}^{\mathbf{V}_0[\dot{G}{\restriction}A_\alpha]}$,
   (b) there is a clopen set $C_\alpha \subseteq {}^\omega 2$ such that
$$\Vdash_\alpha \frac{\mu\big(C_\alpha \cap p(\alpha)\big)}{\mu(C_\alpha)} \geq 1 - \frac{1}{2^{n-j+5}},$$
   where $n = |\mathsf{dom}(p) \setminus \aleph_\omega|$ and $j = |\alpha \cap (\mathsf{dom}(p) \setminus \aleph_\omega)|$.

Lemma 5.16. *D is dense in $\mathcal{P}$.*

Proof. Induction on $\max(\mathsf{dom}(p))$. $\square$

Let $\mathbb{C}$ be the collection of clopen subsets of 2^ω. Represent $\mathbf{C}_{\aleph_{\omega+1}}$ as the colection of functions q such that $\mathsf{dom}(q) \in [\aleph_{\omega+1}]^{<\omega}$ and $q(\alpha) \in \mathbf{C}$ for $\alpha < \aleph_\omega$ and $q(\alpha) \in \mathbb{C}$ for $\alpha \geq \aleph_\omega$.

Note that there is a natural projection π from D to $\mathbf{C}_{\aleph_{\omega+1}}$ defined as
$$\pi(p)(\alpha) = \begin{cases} p(\alpha) & \text{if } \alpha < \aleph_\omega \\ C_\alpha & \text{if } \alpha \geq \aleph_\omega \end{cases}.$$
For a sequence $\bar{p} = \langle p_n : n \in \omega\rangle$ let $\pi(\bar{p}) = \langle \pi(p_n) : n \in \omega\rangle$. Suppose that $\bar{p}$ is such that $\pi(\bar{p}) \in \Delta$, as defined in 5.5. We will define a condition $p_{\bar{p}}$ in the following way; $\mathsf{dom}(p_{\bar{p}}) = \widetilde{\Delta}$, where $\widetilde{\Delta}$ is the root of the Δ-system $\{\mathsf{dom}(p_n) : n \in \omega\}$.

Case 1 $\alpha \in \widetilde{\Delta} \cap \aleph_\omega$.

Let $p_{\bar{p}}(\alpha)$ be the common value of $p_n(\alpha)$ for $n \in \omega$.

Case 2 $\alpha \in \widetilde{\Delta} \setminus \aleph_\omega$.

Work in the model $\mathbf{V} = \mathbf{V}_0[\dot{G}{\restriction}A_\alpha]$ and let $C = \pi(p_n(\alpha))$. Clearly $\mathbf{V} \models C \in \mathbf{B}$. It follows that for some $k \in \omega$ and every $n \in \omega$,
$$\mathbf{V} \models \frac{\mu\big(C \cap p_n(\alpha)\big)}{\mu(C)} \geq 1 - \frac{1}{2^k}.$$

Let $\dot{X}$ be a $\mathbf{B}$-name such that $[\![n \in \dot{X}]\!] = C \cap p_n(\alpha)$. Apply, 5.13 in $\mathbf{V}$, to find a condition $r \in \mathbf{B}, r \geq C$ such that

$$r \Vdash_{\mathbf{B}} \dot{m}^C_{\mathbf{B}}(\dot{X}_{\bar{p}}) \geq 1 - \frac{1}{2^k}.$$

Let $p_{\bar{p}}(\alpha) = r$.

Now we turn our attention to the sequence $\langle \dot{m}^\xi_\alpha : \aleph_\omega \leq \alpha < \aleph_{\omega+1}\rangle$. Let $\{t_\xi : \xi < \aleph_1\}$ be an enumeration of the set T and let $\Delta = \bigcup_{\xi<\aleph_1} \Delta_{t_\xi}$ be the decomposition from 5.7. For $\xi < \aleph_1$ let

$$\Delta^\xi = \{\bar{p} \in [\mathcal{P}]^\omega : \pi(\bar{p}) \in \Delta_{t_\xi}\}.$$

The measure $\dot{m}^\xi_\alpha$ will be first defined on the set

$$\left\{\dot{X}_{\bar{p}} : \bar{p} \in \Delta^\xi \cap [\mathcal{P}_\alpha]^\omega\right\}.$$

We will do it in such a way that for $\bar{p} \in \Delta^\xi \cap [\mathcal{P}_\alpha]^\omega$

$$p_{\bar{p}} \Vdash_\alpha \dot{m}^\xi_\alpha(\dot{X}_{\bar{p}}) > 0,$$

where $p_{\bar{p}}$ is the condition defined above. Next $\dot{m}^\xi_\alpha$ will be extended arbitrarily to the set $\mathbf{P}(\omega) \cap \mathbf{V}_0^{\mathcal{P}_\alpha}$.

Fix $\xi < \aleph_1$ and define $\dot{m}^\xi_\alpha$ as follows:

Case 1. $\alpha = \aleph_\omega$. Consider the family

$$\dot{\mathcal{H}}_\xi = \{\dot{X}_{\bar{p}} : \bar{p} \in \Delta^\xi \cap [\mathcal{P}_{\aleph_\omega}]^\omega,\ p_{\bar{p}} \in \dot{G}_{\mathcal{P}}\}.$$

It is easy to see that $\dot{\mathcal{H}}_\xi$ is a $\mathcal{P}_{\aleph_\omega}$-name for a filter base. Let $\dot{\mathcal{F}}_\xi$ be any $\mathcal{P}$-name for an ultrafilter extending $\dot{\mathcal{H}}_\xi$ and let $\dot{m}^\xi_{\aleph_\omega}$ be the corresponding measure. In other words, for $\dot{X} \in \dot{\mathcal{H}}_\xi$,

$$\Vdash_{\aleph_\omega} \dot{m}^\xi_{\aleph_\omega}(\dot{X}) = 1.$$

Case 2. $\alpha > \aleph_\omega$ and $\mathsf{cf}(\alpha) = \aleph_0$.

Since $\dot{m}^\xi_\alpha$ extends $\bigcup_{\beta<\alpha} \dot{m}^\xi_\beta$, we have to define $\dot{m}^\xi_\alpha$ on the set

$$\left\{\dot{X}_{\bar{p}} : \bar{p} \in \Delta^\xi \cap \left([\mathcal{P}_\alpha]^\omega \setminus \bigcup_{\beta<\alpha}[\mathcal{P}_\beta]^\omega\right)\right\}.$$

Put $\mathcal{A} = \Delta^\xi \cap \left([\mathcal{P}_\alpha]^\omega \setminus \bigcup_{\beta<\alpha}[\mathcal{P}_\beta]^\omega\right)$ and for $\bar{p} \in \mathcal{A}$ let $j = j_{\bar{p}} \in \omega$ be the such that

$$\beta = \sup_{n\in\omega} \alpha^n_{j-1} < \sup_{n\in\omega} \alpha^n_j = \alpha,$$

where α^n_i is the i'th element of $\mathsf{dom}(p_n)$. Consider sequences $\bar{p}^- = \langle p_n \restriction \alpha^n_j : n \in \omega\rangle$ and $\bar{p}^+ = \langle p_n \restriction [\alpha^n_j, \alpha) : n \in \omega\rangle$. Let $\dot{\mathcal{H}}_\xi$ be a $\mathcal{P}_\alpha$-name for the family $\{\dot{X}_{\bar{p}^+} : \bar{p} \in \mathcal{A}\}$. Note that

1. $\Vdash_\alpha \dot{\mathcal{H}}_\xi$ is a filter base,
2. $\forall \dot{X} \in \dot{\mathcal{H}}_\xi\ \forall \beta < \alpha\ \forall \dot{Y} \in [\omega]^\omega \cap \mathbf{V}_0^{\mathcal{P}_\beta}\ \Vdash_\alpha \dot{X} \cap \dot{Y}$ is infinite.

Suppose that $\bar{p} \in \mathcal{A}$ and note that

$$p_{\bar{p}^-} \Vdash_\beta \ \dot{m}^\xi_\beta(\dot{X}_{\bar{p}^-}) = a > 0.$$

By the remarks made above, we can set $\dot{m}_\alpha(\dot{X}_{\bar{p}^+}) = 1$ and $\dot{m}_\alpha(\dot{X}_{\bar{p}}) = a$. Finally note that the value a is forced by $p_{\bar{p}}$.

CASE 3. α is limit and $\mathsf{cf}(\alpha) > \aleph_0$.

Let $\dot{m}^\xi_\alpha = \bigcup_{\beta<\alpha} \dot{m}^\xi_\beta$. This definition is correct since no subsets of ω are added at the step α.

CASE 4. $\alpha = \delta + 1$.

As before we have to define $\dot{m}^\xi_\alpha$ on

$$\left\{\dot{X}_{\bar{p}} : \bar{p} \in \Delta^\xi \cap ([\mathcal{P}_\alpha]^\omega \setminus [\mathcal{P}_\delta]^\omega)\right\}.$$

Set $\mathcal{A} = \Delta^\xi \cap ([\mathcal{P}_\alpha]^\omega \setminus [\mathcal{P}_\delta]^\omega)$ and note that if $\bar{p} \in \mathcal{A}$ then $\delta \in \bigcap_{n\in\omega} \mathsf{dom}(p_n)$. Thus, let C be a clopen set such that $\pi(p_n(\delta)) = C$ for $n \in \omega$. Let $\mathbf{V} = \mathbf{V}_0[\dot{G}{\restriction}A_\delta]$. Find a forcing notion $\mathcal{R}$ such that $\mathcal{P}_\delta = (\mathcal{P}_\delta{\restriction}A_\delta) \star \mathcal{R}$. It follows that $\mathbf{V}_0^{\mathcal{P}_\alpha} = \mathbf{V}_0^{\mathcal{P}_{\delta+1}} = \mathbf{V}^{\mathcal{R}\times\mathbf{B}}$. By the induction hypothesis $m = \dot{m}^\xi_\delta{\restriction}A_\delta$ is a finitely additive measure. In other words $m \in \mathbf{V}$ is a finitely additive measure defined on $\mathbf{P}(\omega) \cap \mathbf{V}$. Clearly $\dot{m}^\xi_\delta$ is an extension of m to $\mathbf{V}^{\mathcal{R}} \cap \mathbf{P}(\omega)$. On the other hand let $\dot{m}^C_{\mathbf{B}}$ be an extension of m to $\mathbf{V}^{\mathbf{B}}$ as given by 5.13. Let $\dot{m}^\xi_\alpha = \dot{m}^\xi_{\delta+1}$ be the common extension of $\dot{m}^\xi_\delta$ and $\dot{m}^C_{\mathbf{B}}$ guaranteed by 5.12. It is clear that $\dot{m}^\xi_\alpha$ has the required properties.

Finally let $\dot{m}^\xi = \bigcup_{\aleph_\omega \le \alpha < \aleph_{\omega+1}} \dot{m}^\xi_\alpha$. Note that each $\dot{m}^\xi$ is a $\mathcal{P}$-name for a finitely additive measure on $\mathbf{P}(\omega) \cap \mathbf{V}_0^{\mathcal{P}}$.

Proof of the Theorem 5.3

We are ready now for the proof of the main theorem. The following lemma gives the lower bound for $\mathsf{cov}(\mathcal{N})$.

Lemma 5.17. $\mathbf{V}_0^{\mathcal{P}} \models \mathsf{cov}(\mathcal{N}) \ge \aleph_\omega$. *In particular,* $[\mathbb{R}]^{<\aleph_\omega} \subseteq \mathsf{COV}(\mathcal{N})$.

Proof. Suppose that $\{H_\alpha : \alpha < \kappa < \aleph_\omega\}$ is a family of measure zero sets in $\mathbf{V}_0^{\mathcal{P}}$. Let N be a master set for $\mathcal{N}$ defined earlier. Without loss of generality we can assume that for some $f_\alpha \in {}^\omega\omega$, $H_\alpha = (N)_{f_\alpha}$, and let $\dot{f}_\alpha$ be a $\mathcal{P}$-name for f_α. As in 5.15, let $K_\alpha \in [\aleph_{\omega+1}]^{\aleph_0} \cap \mathbf{V}_0$ be the set such that $f_\alpha \in \mathbf{V}_0[\dot{G}{\restriction}K_\alpha]$. Find β such that $\bigcup_{\alpha<\kappa} K_\alpha \subseteq A_\beta$. The random real added by $\mathbf{B}^{\mathbf{V}_0[\dot{G}\restriction A_\beta]}$ avoids all null sets coded in $\mathbf{V}_0[\dot{G}{\restriction}A_\beta]$, in particular, all H_α's. $\square$

It remains to be checked that $\mathsf{cov}(\mathcal{N}) \le \aleph_\omega$ in the extension.

Let $X = \{f_\alpha : \alpha < \aleph_\omega\} = \dot{G}{\restriction}\aleph_\omega$ be the sequence of first $\aleph_\omega$ Cohen reals added by $\mathcal{P}$. Our intention is to show that $X \notin \mathsf{COV}(\mathcal{N})$. In fact we will show that

$$\bigcup_{\alpha<\aleph_\omega} (N)_{f_\alpha} = {}^\omega 2,$$

where N is the master set defined in the previous section. That will finish the proof since X is a countable union of sets of smaller size (so they are all in $\mathsf{COV}(\mathcal{N})$) and thus X witnesses that $\mathsf{COV}(\mathcal{N})$ is not a σ-ideal and that $\mathsf{cov}(\mathcal{N}) \le \aleph_\omega$.

Suppose the opposite and let z be such that

$$\mathbf{V}_0^{\mathcal{P}} \models z \notin \bigcup_{\alpha<\aleph_\omega} (N)_{f_\alpha}.$$

Lemma 5.18. *There exists a $\mathcal{P}$-name $\dot{Y}$ for a subset of $\aleph_\omega$ and $\bar{n} \in \omega$ such that*

1. $\Vdash_{\mathcal{P}} \dot{Y} \in [\aleph_\omega]^{\aleph_1}$,
2. $\Vdash_{\mathcal{P}} {}^\omega 2 \setminus \bigcup_{\alpha\in\dot{Y}} \bigcup_{n>\bar{n}} C^n_{f_\alpha(n)}$ *is uncountable.*

Proof. Denote by $\dot{z}$ a $\mathcal{P}$-name for z and let $\delta < \aleph_{\omega+1}$ be the least ordinal such that $\dot{z}$ is a $\mathcal{P}_\delta$-name. We have the following two cases:

Case 1. $\delta = \lambda + 1$ is a successor ordinal.

Suppose first that $\delta > \aleph_\omega$. Work in $\mathbf{V} = {\mathbf{V}_0}^{\mathcal{P}_\lambda}$ and let $\mathbf{B}_\lambda = \mathbf{B}^{\mathbf{V}_0[\dot{G}\restriction A_\lambda]}$. For each $\alpha < \aleph_\omega$ choose $q_\alpha \in \mathbf{B}_\lambda$ and $n_\alpha \in \omega$ such that $\mathbf{V} \models q_\alpha \Vdash_{\mathbf{B}_\lambda} \dot{z} \notin \bigcup_{n>n_\alpha} C^n_{f_\alpha(n)}$. Since $\mathbf{B}_\lambda$ has a dense subset of size $< \aleph_\omega$, we can find $q \in \mathbf{B}_\lambda$ and $\bar{n} \in \omega$ such that the set

$$Y = \{\alpha : q_\alpha = q \;\&\; n_\alpha = \bar{n}\}$$

is uncountable. Consider the set $C = {}^\omega 2 \setminus \bigcup_{\alpha\in Y} \bigcup_{n>\bar{n}} C^n_{f_\alpha(n)}$ in $\mathbf{V}$. Observe that C is a closed set and if it was countable then all its elements would be in $\mathbf{V}$. However, $\mathbf{V}^{\mathbf{B}_\lambda} \models z \in C$ and $z \notin \mathbf{V}$.

If $\delta < \aleph_\omega$ then the argument is identical except that we use $\mathbf{C}$ instead of $\mathbf{B}_\lambda$. In fact one can show that

$$\mathbf{V}_0^{\mathcal{P}} \cap {}^\omega 2 \subseteq \bigcup_{\alpha<\omega_1} (N)_{f_\alpha} \subseteq \bigcup_{\alpha<\aleph_\omega} (N)_{f_\alpha}.$$

Case 2. δ is limit and $\mathsf{cf}(\delta) = \aleph_0$.

In $\mathbf{V}_0^{\mathcal{P}_\delta}$ we can find $\bar{n} \in \omega$ and an uncountable set $Z \subseteq \aleph_\omega$ such that

$$\mathbf{V}_0^{\mathcal{P}_\delta} \models z \notin \bigcup_{\alpha\in Z} \bigcup_{n>\bar{n}} C^n_{f_\alpha(n)}.$$

Let $\dot{Z}$ be a $\mathcal{P}_\delta$-name for Z. Suppose that $G \subseteq \mathcal{P}_\delta$ is a generic filter over $\mathbf{V}_0$. For each $\alpha < \omega_1$ choose $p_\alpha \in \mathcal{P}_\delta \cap G$ and η_α such that $p_\alpha \Vdash_{\mathcal{P}_\delta} \dot{Z}(\alpha) = \eta_\alpha$, where $\dot{Z}(\alpha)$ is a $\mathcal{P}$-name for the α-th element of Z.

There is an uncountable set $I \subseteq \omega_1$, and $\lambda < \delta$ such that $p_\alpha \in \mathcal{P}_\lambda \cap G$ for $\alpha \in I$. Let $Y = \{\eta_\alpha : \alpha \in I\}$ and let $\dot{Y}$ be a $\mathcal{P}_\lambda$-name for Y. As in the previous case, consider the set $C = {}^\omega 2 \setminus \bigcup_{\alpha\in Y} \bigcup_{n>\bar{n}} C^n_{f_{\eta_\alpha}(n)}$ in $\mathbf{V}_0^{\mathcal{P}_\lambda}$. We see that C is uncountable because it contains an element which does not belong to $\mathbf{V}_0^{\mathcal{P}_\lambda}$. □

Find different ordinals $\{\eta_\alpha : \alpha < \omega_1\}$ and conditions $\{p_\alpha : \alpha < \omega_1\} \subseteq \mathcal{P}$ such that $p_\alpha \Vdash_{\mathcal{P}} \eta_\alpha \in \dot{Y}$. Using the Δ-lemma we can assume that there are $\widetilde{k}, \widetilde{l} \in \omega$, $s \in {}^{\widetilde{k}}\omega$ and clopen sets $\{C_j : j \leq \widetilde{l}\}$ such that

1. $\mathsf{dom}(p_\alpha)$ form a Δ-system,
2. $\mathsf{dom}(p_\alpha) = \{\gamma^\alpha_1 < \cdots < \gamma^\alpha_{\tilde{k}} < \aleph_\omega \leq \delta^\alpha_1 < \cdots < \delta^\alpha_{\tilde{l}}\}$,
3. $\forall\alpha\ \forall j \leq \widetilde{k}\ \big(p_\alpha(\gamma^\alpha_j) = s(j)\big)$,
4. for all $j \leq \widetilde{l}$

$$\Vdash_{\alpha_j} \frac{\mu\big(C_j \cap p_\alpha(\delta^\alpha_j)\big)}{\mu(C_j)} \geq 1 - \frac{1}{2^{\tilde{l}-j+5}}.$$

Without loss of generality we can assume that $\eta_\alpha \in \mathsf{dom}(p_\alpha)$. Furthermore we can assume that for some $j_0 \leq \widetilde{k}$, $\eta_\alpha = \gamma^\alpha_{j_0}$ and that $s(j_0) = s^\star$ with $|s^\star| = n^\star$.

Consider the first ω conditions $\bar{p} = \{p_n : n \in \omega\}$ Our next step is to extend p_n's slightly to get a new sequence $\bar{p}^\star$. We will need the following definition.

Definition 5.19. *For a clopen set $C \subseteq {}^\omega 2$ define $\mathsf{supp}(C)$ to be the smallest set $F \subseteq \omega$ such that $C = (C \cap {}^F 2) \times {}^{\omega \setminus F} 2$. In other words, support of C is the set of coordinates that carry information about C.*

Let $K_n = \left\{m : \mathsf{supp}\left(C^{n^\star}_m\right) \subseteq n\right\}$ and let $\{J_n : n \in \omega\}$ be a partition of ω such that $|J_n| = |K_n|$ for each n. Fix a function $o \in {}^\omega\omega$ such that $o``(J_n) = K_n$ for every n. Define

$$p^\star_n = \begin{cases} p_n(\alpha) & \text{if } \alpha \neq \eta_n \\ s^\star {}^\frown \big(n^\star, o(n)\big) & \text{if } \alpha = \eta_n \end{cases}.$$

Observe that there is $\xi < \aleph_1$ such that $\bar{p}^\star = \{p^\star_n : n \in \omega\} \in \Delta^\xi$. This is being witnessed by the $\widetilde{k}, \widetilde{l}$, $s \in {}^k\omega$, clopen sets $\{C_j : j \leq \widetilde{l}\}$ and function g defined as

$$g(i, n) = \begin{cases} s(i) & \text{if } i \leq \widetilde{k},\ i \neq j_0 \\ s^\star {}^\frown \big(n^\star, o(n)\big) & \text{if } i = j_0 \end{cases}.$$

Our goal is to show:

Theorem 5.20. *There exists a condition $p^{\star\star}$ and $\varepsilon > 0$ such that*

$$p^{\star\star} \Vdash_{\mathcal{P}} \exists^\infty n \ \frac{\left|\left\{m \in J_n : p^\star_m \in \dot{G}_{\mathcal{P}}\right\}\right|}{|J_n|} \geq \varepsilon.$$

Before we prove this theorem let us see that the theorem follows readily from it. Recall that in Lemma 5.18 we showed that $\Vdash_{\mathcal{P}} {}^\omega 2 \setminus \bigcup_{\alpha \in \dot{Y}} \bigcup_{n > \bar{n}} C^n_{f_\alpha(n)}$ is uncountable. Since this set is closed, there is a $\mathcal{P}$-name for a tree $\dot{T}$ such that $\Vdash_{\mathcal{P}} \bigcup_{\alpha \in \dot{Y}} \bigcup_{n > \bar{n}} C^n_{f_\alpha(n)} \cap [\dot{T}] = \emptyset$. Let $\dot{Z}_n = \left\{m \in J_n : p^\star_m \in \dot{G}_{\mathcal{P}}\right\}$ for $n \in \omega$. It follows that for every n,

$$p^{\star\star} \Vdash_{\mathcal{P}} \left(\bigcup_{k \in \dot{Z}_n} C^{n^\star}_k\right) \restriction n \cap \dot{T} \restriction n = \emptyset.$$

This is because for a clopen set C and a tree T, if $C \cap [T] = \emptyset$ then $\big(C \restriction \mathsf{supp}(C)\big) \cap \big(T \restriction \mathsf{supp}(C)\big) = \emptyset$. Fix $n \in \omega$ and suppose that $|\dot{T} \restriction n| = m$. The size of the set J_n is equal to $\begin{pmatrix} 2^n \\ 2^{n-n^\star} \end{pmatrix}$. On the other hand the number of sets $C^{n^\star}_k$ which are disjoint with $\dot{T} \restriction n$ is at most $\begin{pmatrix} 2^n - m \\ 2^{n-n^\star} \end{pmatrix}$. Put $2^{-n^\star} = \epsilon$. It follows, (after some calculations), that for some constant $a \geq 1$:

$$\frac{|\dot{Z}_n|}{|J_n|} \leq \frac{\begin{pmatrix} 2^n - m \\ 2^{n-n^\star} \end{pmatrix}}{\begin{pmatrix} 2^n \\ 2^{n-n^\star} \end{pmatrix}} = \prod_{j=1}^{m} \left(1 - \frac{2^{n-n^\star}}{2^n - m + j}\right) \leq a \cdot e^{-\epsilon \cdot m}.$$

Thus

$$\frac{|\dot{Z}_n|}{|J_n|} \leq a \cdot e^{-\epsilon \cdot |\dot{T} \restriction n|}.$$

Since $p^{\star\star} \Vdash_{\mathcal{P}} \limsup_n \dfrac{|\dot{Z}_n|}{|J_n|} \geq \varepsilon$ we get that $p^{\star\star} \Vdash_{\mathcal{P}} \lim_n |\dot{T}{\restriction}n| < \infty$ (the size of $T{\restriction}n$ increases with n). In particular,

$$p^{\star\star} \Vdash_{\mathcal{P}} \dot{T} \text{ is not perfect,}$$

which gives a contradiction.

Proof of the Theorem 5.3: conclusion

In order to finish the proof of 5.3 we have to prove 5.20. We will need one more modification of the sequence $\bar{p}^\star$ and we will require the construction described below.

Lemma 5.21. *Let $\widetilde{\Delta}$ be a finite subset of $\aleph_{\omega+1} \setminus \aleph_\omega$. Suppose that $\{q_i : i \leq N\}$ is a sequence of conditions in $\mathcal{P}$ such that*

1. $\mathsf{dom}(q_i) = \widetilde{\Delta}$,
2. $\forall \alpha \in \widetilde{\Delta}\ \exists a_\alpha\ \forall i \leq N\ \Vdash_\alpha \mu\big(q_i(\alpha)\big) = a_\alpha > 3/4$.

There exists a condition $q^\star$ such that

1. $\mathsf{dom}(q^\star) = \widetilde{\Delta}$,
2. $q^\star \in \mathcal{P}$,
3. $\forall \alpha \in \widetilde{\Delta}\ \Vdash_\alpha \mu\big(q^\star(\alpha)\big) \geq 2a_\alpha - 1$,
4. $q^\star \Vdash_{\mathcal{P}} \{k \leq N : \forall \alpha \in \widetilde{\Delta}\ q^\star{\restriction}\alpha \Vdash_\alpha \dot{x}_\alpha \in q_k(\alpha)\}$ *has at least* $2^{-|\widetilde{\Delta}|} \cdot N \cdot \prod_{\alpha \in \widetilde{\Delta}} a_\alpha$ *elements, where $\dot{x}_\alpha$ is the generic real added by $\dot{G}(\alpha)$.*

Proof. If $\widetilde{\Delta} = \emptyset$, then there is nothing to prove.

Suppose that $\widetilde{\Delta} \neq \emptyset$ and let $\beta = \max(\widetilde{\Delta})$. Let $q'_k = q_k{\restriction}\beta$ for $k \leq N$. Apply the induction hypothesis to get a condition q' such that

1. $\mathsf{dom}(q') = \widetilde{\Delta} \setminus \{\beta\}$,
2. $q' \in \mathcal{P}$,
3. $\forall \alpha \in \widetilde{\Delta} \setminus \{\beta\}\ \Vdash_\alpha \mu\big(q'(\alpha)\big) \geq 2a_\alpha - 1$,
4. $q' \Vdash_{\mathcal{P}} \{k \leq N : \forall \alpha \in \widetilde{\Delta} \setminus \{\beta\}\ q'{\restriction}\alpha \Vdash_\alpha \dot{x}_\alpha \in q_k(\alpha)\}$ has at least $2^{-|\widetilde{\Delta}|} \cdot N \cdot \prod_{\alpha \in \widetilde{\Delta} \setminus \{\beta\}} a_\alpha$ elements.

Let $\dot{W}$ be a $\mathcal{P}$-name for the set

$$\{k \leq N : \forall \alpha \in \widetilde{\Delta} \setminus \{\beta\}\ q'{\restriction}\alpha \Vdash_\alpha \dot{x}_\alpha \in q_k(\alpha)\}.$$

Let $\{W^i : i \leq \ell\}$ be the list of all subsets of N of size at least $2^{-|\widetilde{\Delta}|} \cdot N \cdot \prod_{\alpha \in \widetilde{\Delta} \setminus \{\beta\}} a_\alpha$. Find a maximal antichain $\{q^i : i \leq \ell\}$ below q' such that $q^i \Vdash_{\mathcal{P}} \dot{W} = W^i$ for $i \leq \ell$.

We will need the following easy observation.

Lemma 5.22. *Suppose that $\{A_n : n < N\}$ is a family of subsets of ${}^\omega 2$ of measure $a > 0$. Let*

$$B = \left\{ x \in {}^\omega 2 : x \text{ belongs to at least } \frac{N \cdot a}{2} \text{ sets } A_i \right\}.$$

Then $\mu(B) \geq \max\{a/2, 2a - 1\}$.

Proof. Let χ_{A_i} be the characteristic function of the set A_i for $i \leq N$. It follows that $\int \sum_{i \leq N} \chi_{A_i} = N \cdot a$. On the other hand, estimation of this integral yields,

$$N \cdot \mu(B) + \frac{N \cdot a}{2}\big(1 - \mu(B)\big) \geq N \cdot a$$

and after simple computations we get $\mu(B) \geq \dfrac{a/2}{1-a/2}$. It follows that we get the following estimates:

$$\mu(B) \geq \begin{cases} a/2 & \text{if } a \text{ is close to } 0 \\ 2a-1 & \text{if } a \text{ is close to } 1 \end{cases}.$$

$\square$

Work in $\mathbf{V}^{\mathcal{P}_\beta}$ and for each $i \leq \ell$ apply 5.22 to the family $\{q_k(\beta) : k \in W^i\}$ and obtain a condition $r^i \in \mathbf{B}^{\mathbf{V}_0[\dot{G}\restriction A_\beta]}$ such that

$$r^i \Vdash \{k \in W^i : \dot{x}_\beta \in q_k(\beta)\} \text{ has at least } \frac{|W^i|}{2} \cdot a_\beta \text{ elements,}$$

and $\Vdash_\beta \mu(r^i) \geq 2a_\beta - 1$.

Finally, define $q^\star$ to be a $\mathcal{P}$-name such that for $i \leq \ell$, $q^i \Vdash q^\star(\beta) = r^i$. It is easy to see that $q^\star$ is as required. $\square$

Let $q_k = p^\star_k\restriction\widetilde{\Delta}$, where $\widetilde{\Delta} = \{\alpha_1 < \cdots < \alpha_\ell\}$ is the root of the Δ-system $\big\{\mathsf{dom}\big(p^\star_k\restriction[\aleph_\omega, \aleph_{\omega+1}\big) : k \in \omega\big\}$.

For each n apply 5.21 to the family $\{q_k : k \in J_n\}$ to get a condition $q^\star_n$ such that

1. $\mathsf{dom}(q^\star_n) = \widetilde{\Delta}$,
2. $\forall i \leq \ell \ \Vdash_{\alpha_i} \dfrac{\mu(q^\star(\alpha_i) \cap C_{\alpha_i})}{\mu(C_{\alpha_i})} \geq 2\left(1 - \dfrac{1}{2^{\ell-i+5}}\right) - 1 = \dfrac{1}{2^{\ell-i+4}}$,
3. $q^\star_n \Vdash_{\mathcal{P}} \left|\Big\{k \leq N : \forall \alpha \in \widetilde{\Delta} \ \ (q^\star_n\restriction\alpha \Vdash_\alpha \dot{x}_\alpha \in q_k(\alpha))\Big\}\right| \geq \dfrac{|J_n|}{2^{\ell+1}}$.

Define for $k \in \omega$,

$$p^{\star\star}_k(\alpha) = \begin{cases} p^\star_k(\alpha) \cap q^\star_n & \text{if } \alpha \in \widetilde{\Delta}, \ k \in J_n \\ p^\star_k(\alpha) & \text{otherwise} \end{cases}.$$

Let $\bar{p}^{\star\star} = \{p^{\star\star}_n : n \in \omega\}$. Find $\xi < \aleph_1$ such that $\bar{p}^{\star\star} \in \Delta^\xi$. According to our definitions,

$$p^{\star\star} = p_{\bar{p}^{\star\star}} \Vdash_{\mathcal{P}} \dot{m}^\xi(\dot{X}_{\bar{p}^{\star\star}}) > 0.$$

In particular,

$$p^{\star\star} \Vdash_{\mathcal{P}} \dot{X}_{\bar{p}^{\star\star}} = \{n : p^{\star\star}_n \in \dot{G}_{\mathcal{P}}\} \text{ is infinite.}$$

Let $\varepsilon = 2^{-\ell-1}$ and note that

$$p^{\star\star}_n \Vdash_{\mathcal{P}} \frac{\Big|\{k \in J_n : p^\star_k \in \dot{G}_{\mathcal{P}}\}\Big|}{|J_n|} \geq \varepsilon.$$

It follows that $p^{\star\star}$ is the condition required in 5.20.

Historical remarks Theorem 5.1 was proved by Miller [30]. Better estimates are true (see [7] and [6] or [8]). Theorem 5.2 was proved in [4] (see [8]). Theorem 5.3 is due to Shelah. His paper [41] contains a more a general construction, where in addition $\mathbf{MA}_{\aleph_1}$ holds.

6. Consistency results and counterexamples

This section is devoted to the consistency results involving cardinal invariants of the Cichoń diagram and non-inclusion between the corresponding classes of small sets. We will describe several such constructions in detail.

Suppose that $\mathcal{P}$ is a forcing notion. Let $\mathcal{D}(\mathcal{P})$ denote the family of all dense subsets of elements of $\mathcal{P}$ and $\mathcal{G}(\mathcal{P})$ the family of all filters on $\mathcal{P}$. With $\mathcal{P}$ we can associate the following cardinal invariants:

1. $\mathfrak{ma}(\mathcal{P}) = \min\{|\mathcal{A}| : \mathcal{A} \subseteq \mathcal{D}(\mathcal{P})\ \&\ \neg\exists G \in \mathcal{G}(\mathcal{P})\ \forall D \in \mathcal{A}\ (G \cap D \neq \emptyset)\}$,
2. $\mathfrak{am}(\mathcal{P}) = \min\{|\mathcal{G}| : \mathcal{G} \subseteq \mathcal{G}(\mathcal{P})$ & for every countable sequence $\{D_n : n \in \omega\} \subseteq \mathcal{D}(\mathcal{P})\ \exists G \in \mathcal{G}(\mathcal{P})\ \forall n\ (G \cap D_n \neq \emptyset)\}$.

In other words, $\mathfrak{ma}(\mathcal{P})$ is the size of the smallest family of dense subsets of $\mathcal{P}$ for which there is no filter intersecting all of them and $\mathfrak{am}(\mathcal{P})$ is the size of the smallest family of filters such that for every countable family of dense subsets of $\mathcal{P}$ there is a filter in the family that intersects all of them.

Consider the forcing notions:

- Amoeba forcing $\mathbf{A} = \{U \subseteq {}^\omega 2 : U$ is open and $\mu(U) > 1/2\}$. For $U, V \in \mathbf{A}$, $U \geq V$ if $U \supseteq V$,
- Random real forcing $\mathbf{B} = \{P \subseteq {}^\omega 2 : P$ is a closed set of positive measure $\}$.
- Cohen forcing $\mathbf{C}$.
- Dominating real forcing $\mathbf{D} = \{\langle n, f\rangle : n \in \omega, f \in {}^\omega\omega\}$. For $\langle n, f\rangle, \langle m, g\rangle \in \mathbf{D}$ let $\langle n, f\rangle \geq \langle m, g\rangle$ if $n \geq m$ & $f{\restriction}m = g{\restriction}m$ & $\forall k\ f(k) \geq g(k)$.

We have the following result (see [8] for the proof):

Theorem 6.1. 1. $\mathsf{add}(\mathcal{N}) = \mathfrak{ma}(\mathbf{A})$ *and* $\mathsf{cof}(\mathcal{N}) = \mathfrak{am}(\mathbf{A})$.
2. $\mathsf{cov}(\mathcal{N}) = \mathfrak{ma}(\mathbf{B})$ *and* $\mathsf{non}(\mathcal{N}) = \mathfrak{am}(\mathbf{B})$.
3. $\mathsf{cov}(\mathcal{M}) = \mathfrak{ma}(\mathbf{C})$ *and* $\mathsf{non}(\mathcal{M}) = \mathfrak{am}(\mathbf{C})$.
4. $\mathsf{add}(\mathcal{M}) = \mathfrak{ma}(\mathbf{D})$ *and* $\mathsf{cof}(\mathcal{M}) = \mathfrak{am}(\mathbf{D})$.

This description is particularly well suited to use with the finite support iteration. If $\mathcal{P}$ is a ccc forcing notion having "nice" definition and $\mathcal{P}_\kappa$ is a finite support iteration of $\mathcal{P}$ of length κ then

1. If $\mathbf{V} \models 2^{\aleph_0} = \aleph_1$ then $\mathbf{V}^{\mathcal{P}_{\omega_2}} \models \mathfrak{ma}(\mathcal{P}) = \aleph_2$,
2. If $\mathbf{V} \models 2^{\aleph_0} = \aleph_2$ then $\mathbf{V}^{\mathcal{P}_{\omega_1}} \models \mathfrak{am}(\mathcal{P}) = \aleph_1$.

This example motivates the following definition: a pair of models $\mathbf{V}$ and $\mathbf{V}'$ is dual if

$$\mathbf{V} \models \mathfrak{ma}(\mathcal{P}) = 2^{\aleph_0} \iff \mathbf{V}' \models \mathfrak{am}(\mathcal{P}) < 2^{\aleph_0}.$$

For our purpose we restrict our attention to the coefficients of the Cichoń diagram and define that $\mathbf{V}$ is dual to $\mathbf{V}'$ if all of the following hold:

1. $\mathbf{V} \models \mathsf{cov} = 2^{\aleph_0} \iff \mathbf{V}' \models \mathsf{non} < 2^{\aleph_0}$,
2. $\mathbf{V} \models \mathsf{add} = 2^{\aleph_0} \iff \mathbf{V}' \models \mathsf{cof} < 2^{\aleph_0}$,
3. $\mathbf{V} \models \mathsf{non} = 2^{\aleph_0} \iff \mathbf{V}' \models \mathsf{cov} < 2^{\aleph_0}$,
4. $\mathbf{V} \models \mathsf{cof} = 2^{\aleph_0} \iff \mathbf{V}' \models \mathsf{add} < 2^{\aleph_0}$,
5. $\mathbf{V} \models \mathfrak{b} = 2^{\aleph_0} \iff \mathbf{V}' \models \mathfrak{d} < 2^{\aleph_0}$,
6. $\mathbf{V} \models \mathfrak{d} = 2^{\aleph_0} \iff \mathbf{V}' \models \mathfrak{b} < 2^{\aleph_0}$.

To illustrate this consider the following theories:

$$\mathsf{ZFC} + \mathsf{add}(\mathcal{M}) = \mathsf{cov}(\mathcal{N}) = \aleph_2 + \mathsf{add}(\mathcal{N}) = \aleph_1$$

and

$$\mathsf{ZFC} + \mathsf{cof}(\mathcal{N}) = \aleph_2 + \mathsf{cof}(\mathcal{M}) = \mathsf{non}(\mathcal{N}) = \aleph_1.$$

The first of these models can be obtained by a finite support iteration of $\mathbf{B} \star \mathbf{D}$ of length $\aleph_2$ over a model for CH and the second by iteration of $\mathbf{B} \star \mathbf{D}$ of length $\aleph_1$ over a model for $\neg\mathsf{CH}$. It is clear that $\mathsf{add}(\mathcal{M})$, $\mathsf{cov}(\mathcal{N})$ and $\mathsf{cof}(\mathcal{M})$ and $\mathsf{non}(\mathcal{N})$ have the required values. What is less obvious is that $\mathsf{add}(\mathcal{N}) = \aleph_1$ in the first and $\mathsf{cof}(\mathcal{N}) = \aleph_2$ in the second case. To check that we need a preservation result which ensures that the iteration which we use does not change the value of these invariants. Such theorems were proved in [24], [38] and [8].

We will not study these examples any further because this method has one fundamental weakness: it can give us only some of the models we need. This is because the finite support iteration adds Cohen reals. We will use however the notion of duality outlined above. From now on we will focus on obtaining the models using countable support iteration. To this end we will associate with every cardinal invariant of the Cichoń diagram a proper forcing notion and a preservation theorem as follows:

- $\mathsf{add}(\mathcal{N}) \leftrightsquigarrow$ Amoeba forcing $\mathbf{A}$, preservation of "not adding amoeba reals"
- $\mathsf{cov}(\mathcal{N}) \leftrightsquigarrow$ random real forcing $\mathbf{B}$, preservation of "not adding random reals"
- $\mathsf{cov}(\mathcal{M}) \leftrightsquigarrow$ Cohen forcing $\mathbf{C}$, preservation of "not adding Cohen reals"
- $\mathsf{non}(\mathcal{M}) \leftrightsquigarrow$ forcing $\mathbf{PT}_{f,g}$, preservation of non-meager sets,
- $\mathfrak{b} \leftrightsquigarrow$ Laver forcing $\mathbf{LT}$, preservation of "not adding unbounded reals"
- $\mathfrak{d} \leftrightsquigarrow$ rational perfect set forcing $\mathbf{PT}$, preservation of "not adding dominating reals"
- $2^{\aleph_0} \leftrightsquigarrow$ Sacks forcing $\mathbf{S}$, preservation of Sacks property,
- $\mathsf{cof}(\mathcal{N}) \leftrightsquigarrow$ forcing $\mathbf{S}_2$,
- $\mathsf{non}(\mathcal{N}) \leftrightsquigarrow$ forcing $\mathbf{S}_{g,g^\star}$, preservation of positive outer measure.

We do not assign anything to $\mathsf{add}(\mathcal{M})$ and $\mathsf{cof}(\mathcal{M})$ because they are expressible using the remaining invariants. We refer the reader to [8] for the definitions of all these forcing notions and the formulation of the preservation theorems. We will illustrate the problems with the following examples.

Example 6.2. *Dominating number* $\mathfrak{d}$. *Rational perfect set forcing* $\mathbf{PT}$ *associated with* $\mathfrak{d}$ *is one of the forcing notions that increase* $\mathfrak{d}$ *without affecting other characteristics. The preservation theorem can be stated as follows. We say that a proper forcing notion* $\mathcal{P}$ *is* ${}^\omega\omega$*-bounding if*

$$\forall f \in \mathbf{V}^{\mathcal{P}} \cap {}^\omega\omega \ \exists g \in \mathbf{V} \cap {}^\omega\omega \ \forall n \ \big(f(n) \leq g(n)\big).$$

It is clear that $\mathcal{P}$ *is* ${}^\omega\omega$*-bounding if and only if* $\mathcal{P}$ *preserves dominating families.*

Theorem 6.3. *The countable support iteration of proper* ${}^\omega\omega$*-bounding forcing notions is* ${}^\omega\omega$*-bounding.*

This is the ideal situation – no matter what forcing notion we assign to $\mathsf{cov}(\mathcal{N})$, $\mathsf{non}(\mathcal{M})$, $\mathsf{cof}(\mathcal{N})$ *and* $\mathsf{non}(\mathcal{N})$ *it has to be* ${}^\omega\omega$*-bounding and this property is preserved under countable support iteration iteration.*

Example 6.4. *Covering numbers* $\mathsf{cov}(\mathcal{M})$ *and* $\mathsf{cov}(\mathcal{N})$. *The choice of forcing notions that we assign to these invariants is determined by the Theorem 6.1, it has to be equivalent to Cohen and random real forcing respectively.*

The preservation theorem could be stated as follows (see [23] *or* [8]*).*

Theorem 6.5. *Suppose that $\mathcal{P}_\delta = \lim_{\alpha<\delta} \mathcal{P}_\alpha$ (δ limit) is a countable support iteration of proper forcing notions such that for every $\alpha < \delta$, $\mathcal{P}_\alpha$ does not add random reals. Then $\mathcal{P}_\delta$ does not add random reals.*

The question whether this theorem remain true if we replace words "random" by "Cohen" is open. However, even if the preservation theorem for not adding Cohen reals is true, both result cover only limit stages of the iteration. For the successor steps we do not have an analog of Theorem 6.3, and indeed we can find two ccc forcing notions $\mathcal{P}$ and $\mathcal{Q}$ such that neither $\mathcal{P}$ nor $\mathcal{Q}$ adds random reals but $\mathcal{P} \star \mathcal{Q}$ adds random reals. Similarly for Cohen reals.

These facts impose the following requirements:

- *any iteration of finite length of forcing notions assigned to $\mathfrak{b}$, $\mathsf{non}(\mathcal{M})$, $\mathfrak{d}$, $\mathsf{cov}(\mathcal{M})$, $\mathsf{non}(\mathcal{N})$ and $\mathsf{cof}(\mathcal{N})$ does not add random reals,*
- *iteration of any length of forcing notions assigned to $\mathfrak{b}$, $\mathsf{cov}(\mathcal{N})$, $\mathsf{non}(\mathcal{M})$, $\mathsf{non}(\mathcal{N})$ and $\mathsf{cof}(\mathcal{N})$ does not add Cohen reals.*

It is easy to verify that each of the forcing notions chosen for these invariants have the required properties. However, the reasons why they, for example, do not add Cohen reals are different in each case. Thus, the preservation theorems are often difficult, technical and at the same time not very general.

The full proof that the construction outlined above is possible can be found in [8]. A preservation theorem for not adding Cohen reals that covers the cases we are interested in can be found in [39].

We will take all these constructions for granted and present some applications.

Let us consider the following examples:

Theorem 6.6. *It is consistent with* ZFC *that*

$$\mathfrak{b} = \aleph_2 + \mathsf{cov}(\mathcal{N}) = \mathsf{non}(\mathcal{N}) = \aleph_1.$$

Proof. Recall that for any tree T, $\mathsf{stem}(T)$ is an element of T such that for all $t \in T$, $t \subseteq \mathsf{stem}(T)$ or $\mathsf{stem}(T) \subseteq t$ and for $s \in T$, $\mathsf{succ}_T(s) = \{t : s \subseteq t \;\&\; |t| = |s|+1\}$.

The Laver forcing $\mathbf{LT}$ is the following forcing notion:

$$T \in \mathbf{LT} \iff T \subseteq {}^{<\omega}\omega \text{ is a tree } \& \; \forall s \in T \; \big(|s| \geq \mathsf{stem}(T) \to |\mathsf{succ}_T(s)| = \aleph_0\big).$$

For $T, T' \in \mathbf{LT}$, $T \geq T'$ if $T \subseteq T'$.

Lemma 6.7. 1. $\mathbf{V}^{\mathbf{LT}} \models \mathbf{V} \cap {}^\omega\omega$ *is bounded in* ${}^\omega\omega$.

2. $\mathbf{V}^{\mathbf{LT}} \models \mathbf{V} \cap {}^\omega 2 \notin \mathcal{N}$.
3. $\mathbf{LT}$ *does not add random reals.*

Moreover (2) and (3) hold for the countable support iteration of Laver forcing as well.

Proof. See [8]. □

Let $\mathcal{P}_{\omega_2}$ be a countable support iteration of length $\aleph_2$ the Laver forcing. It follows from Lemma 6.7 that $\mathfrak{b} = \aleph_2$ in $\mathbf{V}^{\mathcal{P}_{\omega_2}}$, while both $\mathsf{cov}(\mathcal{N})$ and $\mathsf{non}(\mathcal{N})$ are equal to $\aleph_1$. □

Theorem 6.8. *It is consistent with* ZFC *that*

$$\mathfrak{d} = \aleph_1 + \mathsf{cov}(\mathcal{N}) = \mathsf{non}(\mathcal{N}) = \aleph_2.$$

Proof. We will use forcing notion $\mathbf{EE}$ defined below rather than $\mathbf{S}_{g,g^\star}$, it has a much simpler definition and has the required properties (the difficulties appear when unbounded reals are added).

The infinitely equal forcing notion $\mathbf{EE}$ is defined as follows: $p \in \mathbf{EE}$ if the following conditions are satisfied:

1. $p : \mathsf{dom}(p) \longrightarrow {}^{<\omega}2$,
2. $p(n) \in {}^{n}2$ for all $n \in \mathsf{dom}(p)$, and
3. $\mathsf{dom}(p) \subseteq \omega,\ |\omega \setminus \mathsf{dom}(p)| = \aleph_0$.

For $p, q \in \mathbf{EE}$ we define $p \geq q$ if $p \supseteq q$.

Lemma 6.9. *Forcing* $\mathbf{EE}$ *has the following properties:*

1. $\mathbf{V}^{\mathcal{P}} \models \mathbf{V} \cap {}^{\omega}2 \in \mathcal{N}$. *In fact,*
$$\forall x \in \mathbf{V} \cap {}^{\omega}2\ \exists^{\infty} n\ (x{\restriction}n = f_G(n)),$$
where f_G *is a generic real.*
2. $\mathcal{P}$ *does not add random reals,*
3. $\mathcal{P}$ *is* ${}^{\omega}\omega$*-bounding.*

Proof. See [8]. $\square$

Let $\left\{\mathcal{P}_\alpha, \dot{\mathcal{Q}}_\alpha : \alpha < \omega_2\right\}$ be a countable support iteration such that for every $\alpha < \omega_2$,

1. $\Vdash_\alpha \dot{\mathcal{Q}}_\alpha \simeq \mathbf{EE}$ if α is even, and
2. $\Vdash_\alpha \dot{\mathcal{Q}}_\alpha \simeq \mathbf{B}$ if α is odd.

Let G be a $\mathcal{P}_{\omega_2}$-generic filter over $\mathbf{V} \models \mathsf{CH}$.

It is clear that $\mathbf{V}[G] \models \mathsf{non}(\mathcal{N}) = \mathsf{cov}(\mathcal{N}) = \aleph_2$. To see that $\mathfrak{d} = \aleph_1$ in the extension note that both forcing notions $\mathbf{B}$ and $\mathbf{EE}$ are ${}^{\omega}\omega$-bounding and use 6.3. $\square$

Now consider the corresponding problem concerning the families of small sets. The question is whether the models constructed for the Cichoń diagram correspond to the sets witnessing the strict inclusion between the corresponding classes of sets.

It is clear that we cannot show that in ZFC alone. For example, it is consistent that $\mathsf{ADD}(\mathcal{N}) = \mathsf{ADD}(\mathcal{M}) = \mathsf{COV}(\mathcal{M}) = [\mathbb{R}]^{\leq\aleph_0}$ (a model for Borel Conjecture, see [8]).

However, the theory $\mathsf{ZFC} + \mathsf{CH}$ provides a sufficiently rich universe in which $<$-results about invariants add, cov, etc in a natural way yield $\subsetneq$ results about ADD, COV, etc.

We will describe here several such constructions in detail. First consider those that involve only forcing notions satisfying ccc.

Theorem 6.10. *(*$\mathsf{ZFC} + \mathsf{CH}$*) There is a set* $X \subseteq \mathbb{R}$ *such that* $X \in \mathsf{D}$ *and* $X \notin \mathsf{NON}(\mathcal{N}) \cup \mathsf{NON}(\mathcal{N})$.

Proof. The construction is canonical. Set the cardinal invariants corresponding to the families that X belongs to to $\aleph_2$ and the other ones to $\aleph_1$. In our case $\mathfrak{d} = \aleph_2$ and $\mathsf{non}(\mathcal{N}) = \mathsf{non}(\mathcal{M}) = \aleph_1$. Now consider the forcing notion that produces the model for the dual setup, i.e. $\mathfrak{b} = \aleph_1$ and $\mathsf{cov}(\mathcal{N}) = \mathsf{cov}(\mathcal{M}) = \aleph_2$. According to our table it is the iteration of Cohen and random forcings, $\mathbf{C} \star \mathbf{B}$. Let $\{M_\alpha : \alpha < \aleph_1\}$ be an increasing sequence of contable submodels of $\mathbf{H}(\lambda)$ such that

1. ${}^{\omega}2 \subseteq \bigcup_{\alpha<\omega_1} M_\alpha$.

2. for ever $\alpha < \omega_1$, $M_{\alpha+1} \models M_\alpha$ is countable,
3. $\{M_\beta : \beta \leq \alpha\} \in M_{\alpha+1}$.

For each α choose a pair $(c_\alpha, r_\alpha) \in M_{\alpha+1}$ such that (c_α, r_α) is $\mathbf{C} \star \mathbf{B}$ over M_α. Note that such a pair will also be generic over M_β for $\beta < \alpha$. Let z_α encode (c_α, r_α) as

$$z_\alpha(n) = \begin{cases} c_\alpha(k) & \text{if } n = 2k \\ r_\alpha(k) & \text{if } n = 2k+1 \end{cases}.$$

Let $X = \{z_\alpha : \alpha < \omega_1\}$. We will show that X has the required properties.

To show that $X \in \mathsf{D}$ fix a Borel function $F : \mathbb{R} \longrightarrow {}^\omega\omega$ and find α_0 such that F is coded in M_{α_0}. Let f be any function which dominates $M_{\alpha_0} \cap {}^\omega\omega$. For any $\alpha < \omega_1$, $F(z_\alpha) \in M_{\alpha_0}^{\mathbf{C}\star\mathbf{B}}$. Since $\mathbf{C} \star \mathbf{B}$ does not add dominating reals it follows that for every α there is a function $g \in M_{\alpha_0} \cap {}^\omega\omega$ such that $g \not\leq^\star F(z_\alpha)$. Since g is dominated by f we conclude that $f \not\leq^\star F(z_\alpha)$ for every $\alpha < \omega_1$.

To see that $X \notin \mathsf{NON}(\mathcal{M}) \cup \mathsf{NON}(\mathcal{N})$ let $Y = \{c_\alpha : \alpha < \omega_1\}$. Observe that Y is a continuous image of X. Moreover, if $F \in M_{\alpha_0}$ is a meager set then $c_\alpha \notin F$ for $\alpha > \alpha_0$ since c_α is a Cohen real over M_{α_0}. The argument that $X \notin \mathsf{NON}(\mathcal{N})$ is identical. □

Observe that the crucial point of the above construction is that the real z_α efined at the step α is generic not only over model M_α but also over models M_β for $\beta < \alpha$. To illustrate this point suppose that $\mathcal{P}$ is a forcing notion, $M \subseteq N$ are two submodels of $\mathbf{H}(\lambda)$ and $\mathcal{P} \in M$. Let $\mathcal{A} \in M$ be a maximal antichain in $\mathcal{P}$. If $\mathcal{P}$ satisfies ccc then $\mathcal{A} \subseteq M$, as a range of a function on ω. If $\mathcal{P}$ is absolutely ccc then $N \models \mathcal{A}$ is an maximal antichain, so a $\mathcal{P}$-generic real over N is also $\mathcal{P}$-generic over M. If $\mathcal{P}$ is not absolutely ccc then we no longer know if $\mathcal{A}$ is a maximal antichain in N. In fact, we do not know if $\mathcal{A}$ is an antichain at all. However, if both M and N are elementary submodels of $\mathbf{H}(\lambda)$, then $N \models \mathcal{A}$ is a maximal antichain. Finally, if $\mathcal{P}$ does not satisfy ccc, then it is no longer true that $\mathcal{A} \subseteq M$, so a $\mathcal{P}$-generic real over $\mathbf{V}$ may not be generic over M. Recall that a condition $p \in \mathcal{P}$ is $(M, \mathcal{P})$-generic if p forces that the above situation does not happen. If for every countable $M \prec \mathbf{H}(\lambda)$ the collection of $(M, \mathcal{P})$-generic conditions is dense in $\mathcal{P}$, then $\mathcal{P}$ is proper.

The following strengthening of properness will allow us to carry out the construction from the proof of 6.10 for non-ccc posets.

Definition 6.11. *Suppose that $\mathcal{P}$ is a forcing notion and $\alpha < \omega_1$ is an ordinal. We say that $\mathcal{P}$ is α-proper if for every sequence $\{M_\beta : \beta \leq \alpha\}$ such that*

1. *for every β, M_β is a countable elementary submodel of $\mathbf{H}(\lambda)$,*
2. *$\{M_\gamma : \gamma \leq \beta\} \in M_{\beta+1}$,*
3. *$M_{\beta+1} \models M_\beta$ is countable,*
4. *$M_\lambda = \bigcup \beta < \lambda M_\beta$ for limit λ,*
5. *$\mathcal{P} \in M_0$,*

and for every $p \in \mathcal{P} \cap M_0$, there exists $q \geq p$ which is $(M_\beta, \mathcal{P})$-generic for $\beta \leq \alpha$.

Definition 6.12. *A forcing notion $\mathcal{P}$ satisfies axiom A if there exists a sequence $\{\leq_n : n \in \omega\}$ of orderings on $\mathcal{P}$ (not necessarily transitive) such that*

1. *if $p \geq_{n+1} q$, then $p \geq_n q$ and $p \geq q$ for $p, q \in \mathcal{P}$,*
2. *if $\langle p_n : n \in \omega \rangle$ is a sequence of conditions such that $p_{n+1} \geq_n p_n$, then there exists $p \in \mathcal{P}$ such that $p \geq_n p_n$ for all n, and*
3. *if $\mathcal{A} \subseteq \mathcal{P}$ is an antichain, then for every $p \in \mathcal{P}$ and $n \in \omega$ there exists $q \geq_n p$ such that $\{r \in \mathcal{A} : q \text{ is compatible with } r\}$ is countable.*

All forcing notions assigned to the cardinal invariants from Cichoń diagram satisfy axiom A.

Lemma 6.13. *If $\mathcal{P}$ satisfies Axiom A then $\mathcal{P}$ is α-proper for every $\alpha < \omega_1$.*

Proof. Induction on α. Let $\langle M_\beta : \beta \leq \alpha \rangle$ be a sequence of models having the required properties. Fix $p \in \mathcal{P} \cap M_0$ and $n \in \omega$. We will find $q \geq_n p$ which is M_β-generic for $\beta \leq \alpha$. If $\alpha = 0$ then it is the usual proof that Axiom A implies properness. If $\alpha = \gamma + 1$ then first find $q' \geq_n p$ which is M_δ-generic for $\delta \leq \gamma$ and then use properness of $\mathcal{P}$ to get $q \geq_n q'$ which is M_α-generic. If α is limit then fix an increasing sequence $\langle \alpha_n : n \in \omega \rangle$ such that $\sup_n \alpha_n = \alpha$. Use the induction hypothesis to find conditions $\{p_k : k \in \omega\}$ such that

1. $p_{k+1} \in M_{\alpha_k + 1}$,
2. p_k is M_γ-generic for $\gamma < \alpha_k$,
3. $p_{k+1} \geq_{n+k} p_k$ for each k.

Let q be such that $q \geq_{n+k} p_k$ for each k, it is the condition we are looking for. □

Theorem 6.14. *($\mathsf{ZFC} + \mathsf{CH}$) There is a set $X \subseteq \mathbb{R}$ such that $X \in \mathsf{B}$ and $X \notin \mathsf{COV}(\mathcal{N}) \cup \mathsf{NON}(\mathcal{N})$.*

Proof. In terms of cardinal invariants the statement of the theorem corresponds to the dual to the model for $\mathfrak{b} = \aleph_2$ and $\mathsf{cov}(\mathcal{N}) = \mathsf{non}(\mathcal{N}) = \aleph_1$, that is the one where $\mathfrak{d} = \aleph_1$ and $\mathsf{cov}(\mathcal{N}) = \mathsf{non}(\mathcal{N}) = \aleph_2$. The set we are looking for is defined using the forcing notion used to construct that model (cf. Theorem 6.8).

Let $\{f_\alpha : \alpha < \omega\}$ be an enumeration of $\mathbb{R}$. Let $\langle M_\alpha : \alpha < \omega_1 \rangle$ be a sequence of countable elementary submodels of $\mathbf{H}(\lambda)$ such that

1. $f_\alpha \in M_\alpha$,
2. $\langle M_\beta : \beta \leq \alpha \rangle \in M_{\alpha+1}$, and $M_{\alpha+1} \models M_\alpha$ is countable,
3. $M_\gamma = \bigcup_{\alpha<\gamma} M_\alpha$ for limit γ.

Note that from (2) it follows that for every $\beta < \alpha$, $M_\alpha \models$ "M_β is countable." Let $\langle e_\alpha, r_\alpha : \alpha < \omega_1 \rangle$ be a sequence of reals such that

1. $e_\alpha, r_\alpha \in M_{\alpha+1}$,
2. e_α is $\mathbf{EE}$-generic over M_β for $\beta \leq \alpha$,
3. r_α is $\mathbf{B}$-generic over $M_\beta[e_\alpha]$ for $\beta \leq \alpha$.

For $\alpha < \omega_1$ define

$$z_\alpha(n) = \begin{cases} e_\alpha(k) & \text{if } n = 2k \\ r_\alpha(k) & \text{if } n = 2k+1 \end{cases}.$$

Let $Z = \{z_\alpha : \alpha < \omega_1\}$.

$Z \notin \mathsf{NON}(\mathcal{N})$. The set $X = \{r_\alpha : \alpha \in \omega_1\}$ is a Borel image of Z. Given $f \in {}^\omega\omega$ find α such that $f = f_\alpha$. Notice that $r_\beta \notin (N)_f$ for $\beta > \alpha$. In particular, no uncountable subset of Z is in $\mathsf{NON}(\mathcal{N})$.

$Z \notin \mathsf{COV}(\mathcal{N})$. Consider the set $Y = \{e_\alpha : \alpha < \omega_1\}$ which is a Borel image of X. Let $\overline{P} = \left\{f \in {}^\omega\left([\omega]^{<\omega}\right) : \forall n \left(f(n) \in {}^n 2\right)\right\}$. Let

$$\widetilde{H} = \{(f, x) : f \in \overline{P}, x \in {}^\omega 2 \ \&\ \exists^\infty n\ x{\restriction}n = f(n)\}.$$

It is easy to see that $\widetilde{H}$ is a Borel set in $\overline{P} \times {}^\omega 2$ and $(\widetilde{H})_f \in \mathcal{N}$ for every f. Suppose that $x \in {}^\omega 2$. Find α such that $x \in M_\alpha$ and note that for $\beta > \alpha$, $x \in (\widetilde{H})_{e_\beta}$. It follows that no uncountable subset of Z is in $\mathsf{COV}(\mathcal{N})$.

$X \in \mathsf{D}$ Let $F : X \longrightarrow {}^\omega\omega$ be a Borel mapping. Find α such that F is coded in M_α. Let $f \in {}^\omega\omega$ be such that for every $g \in M_\alpha \cap {}^\omega\omega$, $g \leq^\star f$. Since M_α is countable, such an f exists. Since both $\mathbf{B}$ and $\mathbf{EE}$ are ${}^\omega\omega$-bounding (so is $\mathbf{B} \star \mathbf{EE}$) for every $\beta > \alpha$, there exists $g \in M_\alpha$ such that $F(z_\alpha) \leq^\star g \leq^\star f$. □

Theorem 6.15. *(*$\mathsf{ZFC} + \mathsf{CH}$*) There is a set $X \subseteq \mathbb{R}$ such that $X \in \mathsf{COV}(\mathcal{N}) \cap \mathsf{NON}(\mathcal{N})$ and $X \notin \mathsf{D}$.*

Proof. Let $\langle M_\alpha : \alpha < \omega_1 \rangle$ be a sequence of countable elementary submodels of $\mathbf{H}(\lambda)$ as in the previous proof.

In this case we use the Laver forcing from Theorem 6.6. The only difference is that in order to ensure that the constructed set belongs to $\mathsf{COV}(\mathcal{N})$ we construct a set of witnesses for that.

Let $\langle l_\alpha, r_\alpha : \alpha < \omega_1 \rangle$ be a sequence of reals such that

1. $l_\alpha, r_\alpha \in M_{\alpha+1}$,
2. l_α is $\mathbf{LT}$-generic over M_β for $\beta \leq \alpha$,
3. r_α is $\mathbf{B}$-generic over $M_\alpha[l_\beta]$ for all $\beta < \omega_1$.

To meet the condition (3) we need the following result:

Theorem 6.16. *Suppose that $N \prec \mathbf{H}(\lambda)$ is a countable model of* ZFC*. Let $S \in N \cap \mathbf{LT}$ and let x be a random real over N. There exists $T \geq S$ such that T is N-generic and $T \Vdash_{\mathbf{LT}} x$ is random over $N[\dot{G}]$.*

Proof. See [24], [33] or [8]. □

Let $X = \{l_\alpha : \alpha < \omega_1\}$. The difference between this and the previous construction is that we define the set of witnesses $\{r_\alpha : \alpha < \omega_1\}$ that $X \in \mathsf{COV}(\mathcal{N})$.

$X \in \mathsf{COV}(\mathcal{N})$. Let $H \subseteq {}^\omega\omega \times {}^\omega 2$ be a Borel set with null sections. Find α such that $H \in M_\alpha$. Note that

$$r_\alpha \notin \bigcup_{\beta < \omega_1} (H)_{l_\beta},$$

since r_α is random over $M_\alpha[l_\beta]$ for all β and $(H)_{l_\beta} \in M_\alpha[l_\beta]$.

$X \in \mathsf{NON}(\mathcal{N})$. Let $F : X \longrightarrow {}^\omega 2$ be a Borel mapping. Find α such that F is coded in M_α. Let $B = \bigcup\{A : A \in \mathcal{N} \cap M_\alpha\}$. Since M_α is countable, B is a null set. By Lemma 6.7(3) for every $\beta > \alpha$, $F(l_\alpha) \in B$.

$X \notin \mathsf{D}$. This is obvious, by Lemma 6.7(1), for every α

$$\forall f \in M_\alpha \cap {}^\omega\omega \ \forall^\infty n \ \big(f(n) < l_\alpha(n)\big).$$

□

The method of constructing counterexamples to the Cichoń diagram described above is very elegant and effective but assumes a rather large body of knowledge involving forcing, preservation theorems etc. We will conclude this section with a sketch of an alternative method of constructing examples of small sets which is also quite general but more direct. Along the way translate the forcing results that we have used into statements about sets of reals.

Suppose that $\mathcal{P}$ is a forcing notion and conditions of $\mathcal{P}$ are sets of reals. Note that all forcing notions associated with The Cichon Diagram are of this form. For a description of much larger class of forcing of that kind see [39].

Let

$$I_{\mathcal{P}} = \left\{X \subseteq \mathbb{R} : \forall p \in \mathcal{P}\ \exists q \geq p\ \left(q \cap X = \emptyset\right)\right\}.$$

The following lemma lists the obvious observations about $I_{\mathcal{P}}$.

Lemma 6.17. 1. *$I_{\mathcal{P}}$ is an ideal,*
2. *$X \in I_{\mathcal{P}}$ iff there exists a maximal antichain $\mathcal{A} \subseteq \mathcal{P}$ such that $X \cap \bigcup \mathcal{A} = \emptyset$,*
3. *$\forall p \in \mathcal{P}\ (p \notin I_{\mathcal{P}})$.*

Suppose that $\mathcal{P}$ is a forcing notion satisfying Axiom A. Let

$$I^{\omega}_{\mathcal{P}} = \left\{X \subseteq \mathbb{R} : \forall p \in \mathcal{P}\ \forall n \in \omega\ \exists q \geq_n p\ \left(q \cap X = \emptyset\right)\right\}.$$

Note that $I^{\omega}_{\mathcal{P}}$ is a σ-ideal contained in $I_{\mathcal{P}}$.

If $\mathcal{P}$ satisfies ccc, then we can witness that $\mathcal{P}$ satisfies Axiom A by putting $p \leq_0 q$ if $p \leq q$, and for $n > 0$, $p \leq_n q$ if $p = q$. In this case $I^{\omega}_{\mathcal{P}} = \emptyset$. However, for non-ccc forcings as well as some ccc posets (like random real algebra $\mathbf{B}$) we can define $\leq_n$'s in such a way that $I^{\omega}_{\mathcal{P}} = I_{\mathcal{P}}$.

First we will describe how to translate the forcing theorems.

Lemma 6.18. *Suppose that $\mathcal{P}$ is a forcing notion such that*

1. *$\mathcal{P}$ is proper,*
2. *for every $\mathbf{V}$-generic filter $G \subseteq \mathcal{P}$ there exists a real x_G such that $\mathbf{V}[G] = \mathbf{V}[x_G]$,*
3. *conditions of $\mathcal{P}$ are Borel sets of reals, ordered by inclusion,*
4. *every countable antichain in $\mathcal{P}$ can be represented by a countable family of pairwise disjoint elements of $\mathcal{P}$.*

Then for every $\mathcal{P}$-name $\dot{x}$ such that $\Vdash_{\mathcal{P}} \dot{x} \in {}^{\omega}2$ and $p \in \mathcal{P}$ there exists a Borel function $F \in \mathbf{V}$, $F : {}^{\omega}2 \longrightarrow {}^{\omega}2$ and a $q \geq p$ such that $q \Vdash_{\mathcal{P}} \dot{x} = F(x_{\dot{G}})$.

Proof. Fix $\dot{x}$ and let $\mathcal{A}_n$ be a maximal antichain of conditions deciding $\dot{x}{\restriction}n$. Use properness to find $q \geq p$ such that each $\mathcal{A}'_n = \{r \in \mathcal{A}_n : r \text{ is compatible with } q\}$ is countable. By the assumption we can assume that elements of $\mathcal{A}'_n$ are pairwise disjoint. Define $F_n : q \longrightarrow 2^n$ as

$$F_n(x) = s \text{ if } x \in r \in \mathcal{A}'_n \text{ and } r \Vdash_{\mathcal{P}} \dot{x}{\restriction}n = s.$$

Note that $F = \lim_n F_n$ is the function we are looking for. $\square$

Let $\mathcal{P}$ be a forcing notion satisfying the assumptions of the above lemma.

- $\mathcal{P}$ does not add random reals if for every $\mathcal{P}$-name $\dot{x}$ for an element of ${}^{\omega}2$ and every $p \in \mathcal{P}$ there is $q \geq p$ and $H \in \mathbf{V} \cap \mathcal{N}$ such that $q \Vdash_{\mathcal{P}} \dot{x} \in H$.
- $\mathcal{P}$ is ${}^{\omega}\omega$-bounding if for every $\mathcal{P}$-name $\dot{f}$ for an element of ${}^{\omega}\omega$ and every $p \in \mathcal{P}$ there is $q \geq p$ and $g \in \mathbf{V} \cap {}^{\omega}\omega$ such that $q \Vdash_{\mathcal{P}} \dot{f} \leq^{\star} g$.
- $\mathcal{P}$ preserves outer measure if for every set of positive outer measure $X \subseteq {}^{\omega}2$, $X \in \mathbf{V}$ and every $\dot{F}$, a $\mathcal{P}$-name for a Borel function from ${}^{\omega}2$ to ${}^{\omega}\omega$ and $p \in \mathcal{P}$ there is $q \geq p$ such that $q \Vdash_{\mathcal{P}} X \setminus (N)_{\dot{F}(x_{\dot{G}})}) \neq \emptyset$.

These statements translate as:

- (not adding random reals) For every Borel fuction $F : {}^{\omega}2 \longrightarrow {}^{\omega}2$ and $p \in \mathcal{P}$ there is a set $H \in \mathcal{N}$, $q \geq p$ and $A \in I_{\mathcal{P}}$ such that $F``(q \setminus A) \subseteq H$.
- ($\mathcal{P}$ is ${}^{\omega}\omega$-bounding) For every Borel fuction $F : {}^{\omega}2 \longrightarrow {}^{\omega}\omega$ and $p \in \mathcal{P}$ there is a function $f \in {}^{\omega}\omega$, $q \geq p$ and $A \in I_{\mathcal{P}}$ such that $F``(q \setminus A) \leq^{\star} f$.

- ($\mathcal{P}$ preserves outer measure) for every set of positive outer measure $X \subseteq {}^{\omega}2$, and every Borel function $F : {}^{\omega}2 \longrightarrow {}^{\omega}\omega$ and $p \in \mathcal{P}$ there is $q \geq p$ and $A \in I_{\mathcal{P}}$ such that $X \setminus \bigcup_{x \in q \setminus A} (N)_{F(x)} \neq \emptyset$.

If in addition $\mathcal{P}$ satisfies axiom A and $I_{\mathcal{P}} = I^{\omega}_{\mathcal{P}}$, then we can put $A = \emptyset$.

Second proof of 6.14. For $p, q \in \mathbf{EE}$ and $n \in \omega$ we define $p \geq_n q$ if $p \geq q$ and first n elements of $\omega \setminus \mathsf{dom}(p)$ and $\omega \setminus \mathsf{dom}(q)$ are the same.

For $p, q \in \mathbf{B}$ and $n \in \omega$ let $p \geq_n q$ if $p \geq q$ and $\mu(q \setminus p) \leq 2^{-n} \cdot \mu(q)$.

The forcing notions $\mathbf{EE}$, $\mathbf{B}$ (and the remaining ones as well) can be represented as the collections of perfect subsets of ${}^{\omega}2$ (or ${}^{\omega}\omega$). This is not critical for the construction, but it makes it more natural.

In case of $\mathbf{EE}$ for $n \in \omega$ let $k_n = 2^{n+1} - 1$. Consider sets $P \subseteq {}^{\omega}2$ of form $\bigcap_{n \in \omega} [C_n]$, where $\{C_n : n \in \omega\}$ satisfies the following conditions:

1. $C_n \subseteq {}^{[k_n, k_{n+1})}2$,
2. for every n, $|C_n| = 1$ or $|C_n| = 2^n$ (so $C_n = {}^{[k_n, k_{n+1})}2$),
3. $\exists^{\infty} n \ |C_n| = 2^n$.

It is clear that every condition $p \in \mathbf{EE}$ corresponds to a set P as above and vice versa. Therefore from now on we identify $\mathbf{EE}$ with these sets.

Let $\mathbf{B} \star \mathbf{EE}$ be the collection of subsets $H \subseteq 2^{\omega} \times 2^{\omega}$ such that

1. H is Borel and $\mathsf{dom}(H) = \{x : (H)_x \neq \emptyset\} \in \mathbf{B}$,
2. $\forall x \ \left((H)_x \neq \emptyset \rightarrow (H)_x \in \mathbf{EE}\right)$.

The elements of $\mathbf{B} \star \mathbf{EE}$ are $\mathbf{B}$-names for the elements of $\mathbf{EE}$. Thus, the set $\mathbf{B} \star \mathbf{EE}$ indeed corresponds to the iteration of $\mathbf{B}$ and $\mathbf{EE}$. For $H_1, H_2 \in \mathbf{B} \star \mathbf{EE}$ and $n \in \omega$ let $H_1 \geq H_2$ mean that

1. $\mathsf{dom}(H_1) \geq_n \mathsf{dom}(H_2)$,
2. $\forall x \in \mathsf{dom}(H_1) \ \big((H_1)_x \geq_n (H_2)_x\big)$.

Note that $\geq_n$ on $\mathbf{B} \star \mathbf{EE}$ witnesses that it satisfies Axiom A.

Let $\langle x_\alpha : \alpha < \omega_1 \rangle$ be an enumeration of ${}^{\omega}2$, and $\langle F_\alpha : \alpha < \omega_1 \rangle$ of $\mathsf{BOREL}({}^{\omega}2 \times {}^{\omega}2, {}^{\omega}\omega)$, and $\langle f_\alpha : \alpha < \omega_1 \rangle$ an enumeration of ${}^{\omega}\omega$. We will build an ω_1-tree $\mathbf{A}$ of elements of $\mathbf{B} \star \mathbf{EE}$. Let $\mathbf{A}_\alpha$ denote the α-th level of $\mathbf{A}$. The tree $\mathbf{A}$ satisfies the following inductive conditions:

1. $\forall \beta > \alpha \ \forall n \ \forall H \in \mathbf{A}_\alpha \ \exists H' \in \mathbf{A}_\beta \ \big(H' \geq_n H\big)$.
2. $\exists f \in {}^{\omega}\omega \ \forall H \in \mathbf{A}_{\alpha+1} \ \big(F_\alpha``(H) \leq^{\star} f\big)$,
3. $\forall H \in \mathbf{A}_{\alpha+1} \ \big(\mathsf{dom}(H) \cap (N)_{f_\alpha} = \emptyset\big)$,
4. $\forall H \in \mathbf{A}_{\alpha+1} \ \forall x \in \mathsf{dom}(H) \ \exists^{\infty} n \ \Big(|C^x_n| = 1 \ \& \ C^x_n \subseteq x_\alpha\Big)$, where $(H)_x = \bigcap_n [C^x_n]$.

Case 1. $\alpha = \beta + 1$. We will describe how to build a set of immediate successors of an element $H \in \mathbf{A}_\beta$. Given $H \in \mathbf{A}_\beta$ and $n \in \omega$ find $H'_n \geq_n H$ satisfying conditions (3) and (4). By further shinking we can ensure that (2) holds as well. Condition (2) is just the statement that the iteration of $\mathbf{EE}$ and $\mathbf{B}$ is ${}^{\omega}\omega$-bounding.

Case 2. α is limit. Suppose that $H \in \mathbf{A}_{\beta_0}$ for some $\beta_0 \in \omega$ and that $n \in \omega$ is given. Fix an increasing sequence $\langle \beta_k : k \in \omega \rangle$ such that $\beta_k \rightarrow \alpha$. Choose a sequence $\langle H_k : k \in \omega \rangle$ such that

1. $H_0 = H_1 = \cdots = H_n = H$,

2. for $k \geq 0$, $H_{n+k+1} \geq_n H_{n+k}$,
3. $H_{k+n} \in \mathbf{A}_{\beta_k}$.

Use Axiom A to find H' such that $H' \geq_k H_k$. Level $\mathbf{A}_\alpha$ will consist of elements selected in this way.

Let $X = \{(x_\alpha, y_\alpha) : \alpha < \omega_1\}$ be a selector from elements of $\mathbf{A}$. Note that $\pi_1(X) = \{x_\alpha : \alpha < \omega_1\} \notin \mathsf{NON}(\mathcal{N})$ (by (3)), $\pi_2(X) = \{y_\alpha : \alpha < \omega_1\} \notin \mathsf{COV}(\mathcal{N})$ (by (4)) and $X \in \mathsf{D}$ (by (2)). $\square$

Now let us look at the set constructed in Theorem 6.15.

Second proof of 6.15. For every $T \in \mathbf{LT}$ and $s \in {}^{<\omega}\omega$ define a node $T(s)$ in the following way: $T(\emptyset) = \mathsf{stem}(T)$ and for $n \in \omega$ let $T(s^\frown n)$ be the n-th element of $\mathsf{succ}_T\big(T(s)\big)$.

For $T, T' \in \mathbf{LT}$ and $n \in \omega$ define $T \geq_n T'$ if $T \geq T'$ & $\forall s \in n^{\leq n}\ \big(T(s) = T'(s)\big)$. In particular, $T \geq_0 T'$ is equivalent to $T \geq T'$ and $\mathsf{stem}(T) = \mathsf{stem}(T')$. It is easy to check that Laver forcing satisfies Axiom A.

Suppose that

1. $\langle f_\alpha : \alpha < \omega_1\rangle$ is an enumeration of ${}^\omega\omega$,
2. $\langle F_\alpha : \alpha < \omega_1\rangle$ is an enumeration of $\mathsf{BOREL}({}^\omega\omega, {}^\omega 2)$,
3. $\langle G_\alpha : \alpha < \omega_1\rangle$ is an enumeration of $\mathsf{BOREL}({}^\omega\omega, {}^\omega\omega)$.

We build an ω_1-tree $\mathbf{A}$ satisfying the following inductive conditions:

1. $\forall \beta > \alpha\ \forall n\ \forall T \in \mathbf{A}_\alpha\ \exists S \in \mathbf{A}_\beta\ \big(S \geq_n T\big)$,
2. $\forall T \in \mathbf{A}_{\alpha+1}\ \forall x \in [T]\ f_\alpha \leq^\star x$, ($\mathbf{LT}$ adds a dominating real (6.7(1)))
3. for every $T \in \mathbf{A}_{\alpha+1}$, F_α“(T) has measure zero, ($\mathbf{LT}$ does not add random reals (6.7(3)))
4. $\forall T \in \mathbf{A}_{\alpha+1}\ \Big({}^\omega 2 \setminus \bigcup_{x\in[T]}(N)_{G_\alpha(x)} \text{ is uncountable }\Big)$. ($\mathbf{LT}$ preserves outer measure (6.7(2))),

Next we want to chose a selector X from elements of $\mathbf{A}$. Condition (2) will guarantee that $X \notin \mathsf{D}$ and (3) that $X \in \mathsf{NON}(\mathcal{N})$. Unfortunately (4) does not suffice to show that $X \in \mathsf{COV}(\mathcal{N})$. It is conceivable that $2^\omega = \bigcup_{T\in\mathbf{A}_{\alpha+1}} \bigcup_{x\in[T]}(N)_{G_\alpha(x)}$, because $\mathsf{COV}(\mathcal{N})$ is not a σ-ideal. Therefore we need stronger property:

4’. For every Borel function $F : {}^\omega\omega \longrightarrow {}^\omega\omega$ and a sequence $\{T_n : n \in \omega\}$ of conditions in $\mathbf{LT}$ there exists an uncountable set $Y \subseteq 2^\omega$ such that for each $x \in Y$we can find sequence $\{S^n_k : n, k \in \omega\}$ such that $S^n_k \geq_k T_n$ and $y \notin \bigcup_{n,k}\bigcup_{x\in[S^n_k]}(N)_{F(n)}$.

Property (4’) is a translation of 6.16.

Now we construct X along with $\mathbf{A}$. On the step α we have X_α and $\mathbf{A}_\alpha$. Let $\mathbf{A}_\alpha = \{T_n : n \in \omega\}$ and pick $y \notin X_\alpha$ together with $\{S^n_k : n, k \in \omega\} = \mathbf{A}_{\alpha+1}$ as in (4’).

$\square$

Historical remarks Parts (1) and (4) of Theorem 6.1 are due to Truss [52] and [51] and parts (2) and (3) to Solovay [47]. Theorem 6.3 and other preservation results are due to Shelah [42]. Various presentations of these results appear in [19], [23] and most generally in [44]. Models for the Cichoń diagram were constructed by Miller in [29], more in [24] and the latest ones in [9]. Theorem 6.7(2) is due to Judah

and Shelah ([24]), the remaining parts are due to Laver [27]. The forcing **EE** and Lemma 6.9 are due to Miller [29]. Brendle [11] constructed the counterexamples for the $\subsetneq$ for the families of small sets. This type of constructions were already considered in [17]. The technique of "Aronszajn tree of perfect sets" was invented by Todorcevic (see [18]). Theorem 6.16 is due to Judah and Shelah.

Acknowledgement: I would like to thank Andrzej Rosłanowski for careful reading early versions of this article and many helpful suggestions and remarks.

References

[1] Joan Bagaria and Haim Judah. Amoeba forcing, Suslin absoluteness and additivity of measure. In *Set Theory of the continuum*, MSRI Proceedings, pages 155–173. Springer Verlag, 1992.

[2] Tomek Bartoszyński. Additivity of measure implies additivity of category. *Transactions of the American Mathematical Society*, 281(1):209–213, 1984.

[3] Tomek Bartoszyński. Combinatorial aspects of measure and category. *Fundamenta Mathematicae*, 127(3):225–239, 1987.

[4] Tomek Bartoszyński. On covering of real line by null sets. *Pacific Journal of Mathematics*, 131(1):1–12, 1988.

[5] Tomek Bartoszyński, Martin Goldstern, Haim Judah, and Saharon Shelah. All meager filters may be null. *Proceedings of the American Mathematical Society*, 117(2):515–521, 1993.

[6] Tomek Bartoszyński and Haim Judah. On the cofinality of the smallest covering of the real line by meager sets. *The Journal of Symbolic Logic*, 54(3):828–832, 1989.

[7] Tomek Bartoszyński and Haim Judah. On the smallest covering of the real line by meager sets II. *Proceedings of the American Mathematical Society*, 123(6):1879–1885, 1995.

[8] Tomek Bartoszyński and Haim Judah. *Set Theory: on the structure of the real line.* A.K. Peters, 1995.

[9] Tomek Bartoszyński, Haim Judah, and Saharon Shelah. The Cichoń Diagram. *The Journal of Symbolic Logic*, 58(2):401–423, 1993.

[10] Andreas Blass. Reductions between cardinal characteristics of the continuum. In Tomek Bartoszyński and Marion Scheepers, editors, *Set theory: BEST I–III*, volume 192 of *Contemporary Mathematics*, pages 31–50. American Mathematical Society, 1996.

[11] Jorg Brendle. Generic constructions of small sets of reals. *Topology Appl.*, 71(2):125–147, 1996.

[12] J.P.R Christensen. Necessary and sufficient conditions for the measurability of certain sets of closed subsets. *Math. Ann.*, 200:189–193, 1973.

[13] Jacek Cichoń. On two-cardinal properties of ideals. *Transactions of the American Mathematical Society*, 314(2):693–708, 1989.

[14] David H. Fremlin. Cichoń's diagram, 1984. presented at the Séminaire Initiation à l'Analyse, G. Choquet, M. Rogalski, J. Saint Raymond, at the Université Pierre et Marie Curie, Paris, 23e année.

[15] David H. Fremlin. Families of compact sets and Tukey's ordering. *Atti del Seminario Matematico e Fisico dell'Universita di Modena*, 39(1):29–50, 1991.

[16] David H. Fremlin. The partial orderings in measure theory and Tukey ordering. *Note di Matematica*, XI:177–214, 1991.

[17] David H. Fremlin and Arnold W. Miller. On some properties of Hurewicz, Menger, and Rothberger. *Fundamenta Mathematicae*, 129(1):17–33, 1988.

[18] Fred Galvin and Arnold W. Miller. γ-sets and other singular sets of real numbers. *Topology and its Applications*, 17(2):145–155, 1984.

[19] Martin Goldstern. Tools for your forcing constructions. In Haim Judah, editor, *Set theory of the reals*, Israel Mathematical Conference Proceedings, pages 305–360. Bar Ilan University, 1992.

[20] J.R. Isbell. The category of cofinal types II. *Transactions of the American Mathematical Society*, 116:394–416, 1965.

[21] H. Judah, W. Just, and Woodin H., editors. *A weak version of AT which follows from OCA*, New York, 1992. Springer.

[22] Haim Judah and Miroslav Repický. Amoeba reals. *The Journal of Symbolic Logic*, 60(4):1168–1185, 1995.
[23] Haim Judah and Miroslav Repický. No random reals in countable support iterations. *Israel Journal of Mathematics*, 92(1–3):349–359, 1995.
[24] Haim Judah and Saharon Shelah. The Kunen-Miller chart (Lebesgue measure, the Baire property, Laver reals and preservation theorems for forcing). *The Journal of Symbolic Logic*, 55(3):909–927, 1990.
[25] A. S. Kechris, A. Louveau, and W. H. Woodin. The structure of σ-ideals of compact sets. *Trans. Amer. Math. Soc.*, 301:263–288, 1987.
[26] Alexander Kechris. *Classical Descriptive Set Theory*, volume 156 of *Graduate Texts in Mathematics*. Springer Verlag, 1995.
[27] Richard Laver. On the consistency of Borel's conjecture. *Acta-Math.*, 137(3-4):151–169, 1976.
[28] Alain Louveau and Boban Velickovic. Analytic ideals and cofinal types, 1996. preprint.
[29] Arnold W. Miller. Some properties of measure and category. *Transactions of the American Mathematical Society*, 266(1):93–114, 1981.
[30] Arnold W. Miller. The Baire category theorem and cardinals of countable cofinality. *The Journal of Symbolic Logic*, 47(2):275–288, 1982.
[31] John C. Oxtoby. *Measure and category*, volume 2 of *Graduate Texts in Mathematics*. Springer Verlag, New York-Berlin, 1980.
[32] Janusz Pawlikowski. Lebesgue measurability implies Baire property. *Bulletin des Sciences Mathematiques. 2e Serie*, 109(3):321–324, 1985.
[33] Janusz Pawlikowski. Laver's forcing and outer measure. In Tomek Bartoszyński and Marion Scheepers, editors, *Set theory: BEST I–III*, volume 192 of *Contemporary Mathematics*. American Mathematical Society, 1996.
[34] Janusz Pawlikowski and Ireneusz Recław. Parametrized Cichoń's diagram and small sets. *Fundamenta Mathematicae*, 147(2):135–155, 1995.
[35] Jean Raisonnier and Jacques Stern. The strength of measurability hypotheses. *Israel Journal of Mathematics*, 50(4):337–349, 1985.
[36] J. Saint Raymond. Caracterisations d'espaces Polnais, d'apres des travaux recents de J.P.R Christensen et d., 1971–73.
[37] Ireneusz Recław. Cichon's Diagram and Continuum Hypothesis. preprint.
[38] Miroslav Repický. Properties of forcing preserved by finite support iterations. *Comment. Math. Univ. Carolin*, 32:95–103, 1991.
[39] Andrzej Rosłanowski and Saharon Shelah. Norms on possibilities i: forcing with trees and creatures. *Memoirs of the American Mathematical Society*, 1998.
[40] Fritz Rothberger. Eine aquivalenz zwishen der kontinuumhypothese under der Existenz der Lusinschen und Sierpinschishen mengen. *Fundamenta Mathematicae*, 30, 1938.
[41] Saharon Shelah. Covering of the null ideal may have countable cofinality. preprint, Sh592.
[42] Saharon Shelah. *Proper Forcing*, volume 940 of *Lecture Notes in Mathematics*. Springer-Verlag, 1982.
[43] Saharon Shelah. Can you take Solovay inaccessible away? *Israel J. of Math.*, 48:1–47, 1984.
[44] Saharon Shelah. *Proper and Improper Forcing*. Perspectives in Logic. Springer-Verlag, 1998.
[45] Slawomir Solecki. Analytic ideals. *Bulletin of Symbolic Logic*, 2(3):339–348, 1996.
[46] Slawomir Solecki. Analytic ideals and their applications, 1997. preprint.
[47] Robert M. Solovay. A model of set-theory in which every set of reals is Lebesgue measurable. *Annals of Mathematics*, 92(2):1–56, 1970.
[48] Michel Talagrand. Compacts de fonctions mesurables et filtres non mesurables. *Studia Mathematica*, 67(1):13–43, 1980.
[49] Stevo Todorcevic. Every analytic p-ideal is Tukey reducible to ℓ^1. Note of June 1996.
[50] Stevo Todorcevic. Analytic gaps. *Fundamenta Mathematicae*, 147:55–66, 1995.
[51] J. K. Truss. Connections between different amoeba algebras. *Fundamenta Mathematicae*, 130(2):137–155, 1988.
[52] John Truss. Sets having calibre $\aleph_1$. In *Logic Colloquium 76 (Oxford, 1976)*, volume 87 of *Studies in Logic and Found. Math.*, pages 595–612, Amsterdam, 1977. North Holland.
[53] J.W. Tukey. *Convergence and uniformity in topology*. Annals of Mathematics Studies. Princeton University Press, 1940.

[54] Peter Vojtáš. Generalized Galois-Tukey connections between explicit relations on classical objects of real analysis. In Haim Judah, editor, *Set theory of the reals*, Israel Mathematical Conference Proceedings, pages 619–643. Bar Ilan University, 1992.

Department of Mathematics and Computer Science, Boise State University, Boise, Idaho 83725 U.S.A.

E-mail address: `tomek@math.idbsu.edu, http://math.idbsu.edu/~tomek`